Chapter 4
Non-Linear Oscillations and Chaos

Non-Linear Differential Equations

Up to now we have considered differential equations with terms that are proportional to the acceleration, the velocity, and the position:

$$m\ddot{x} + f(\dot{x}) + g(x) = h(t)$$

Such an equation is called a **linear differential equation**. Although many important applications can be described in terms of a linear differential equation, many other applications require the inclusion of terms that are higher than linear. In that case, the system is called **non-linear**. Differential equations can become non-linear if for example:

- The restoring force is not proportional to the displacement x.
- The damping force is not proportional to the velocity v.

An example of the first case is the pendulum. The pendulum will exhibit simple-harmonic motion if the angular displacement is small. In that case, the restoring force, which is proportional of $\sin\theta$ is approximately proportional to θ. However, if the angular displacement is large, $\sin\theta$ is no longer proportional to θ and the system will no longer exhibit simple harmonic motion.

Systems in which the force is non-linear can be divided in to two groups:

1. **Systems for which the force is symmetric around the equilibrium position** (see Figure 1). The non-linear behavior, which is added to the linear behavior, must be proportional to x^3 since terms proportional to x^2 would reduce the magnitude of the restoring force on one side, while increases its magnitude on the opposite side:

$$F(x) = -kx + \varepsilon x^3$$

   If $\varepsilon > 0$, the magnitude of the force is decreased compared to the linear situation. Such a system is called a soft system. If $\varepsilon < 0$, the magnitude of the force is increased compared to the linear situation. Such a system is called a hard system.

2. **Systems for which the force is asymmetric around the equilibrium position** (see Figure 2). The non-linear behavior, which is added to the linear behavior, must be proportional to x^2 since terms proportional to x^3 would reduce the magnitude of the restoring force on both sides:

$$F(x) = -kx + \lambda x^2$$

If $\lambda > 0$, the force is increased compared to the linear situation. If $\lambda < 0$, the force is decreased compared to the linear situation. In both cases, the system is hard on one side of the equilibrium position, and soft on the opposing side.

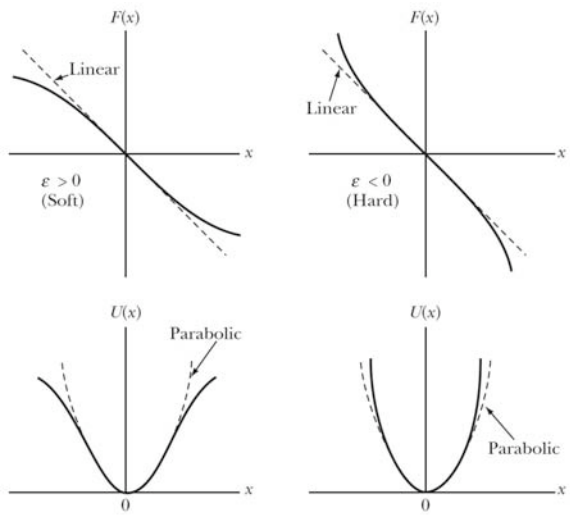


Figure 1. Examples of forces that are symmetric around the equilibrium position.

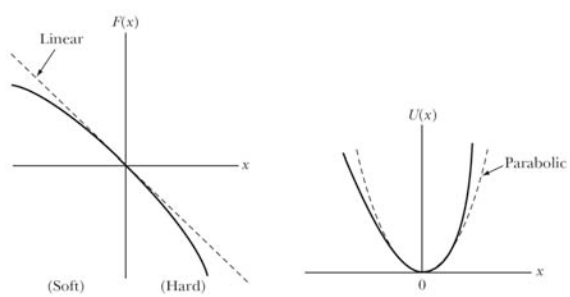


Figure 2. An example of a force that is asymmetric around the equilibrium position.

<u>Phase Diagrams for Non-Linear Systems</u>

If the potential energy of the linear system is known, it is easy to construct the phase diagram. The procedure used to construct the phase diagram relies on conservation of energy:

$$E = U(x) + \frac{1}{2}m\dot{x}^2$$

or

$$\dot{x} = \frac{2}{m}\sqrt{E - U(x)}$$

Examples of phase diagram for stable and unstable equilibrium positions are shown in Figure 3. The curves of constant energy are closed when we are dealing with a stable equilibrium, while they are not closed when we are dealing with unstable equilibrium.

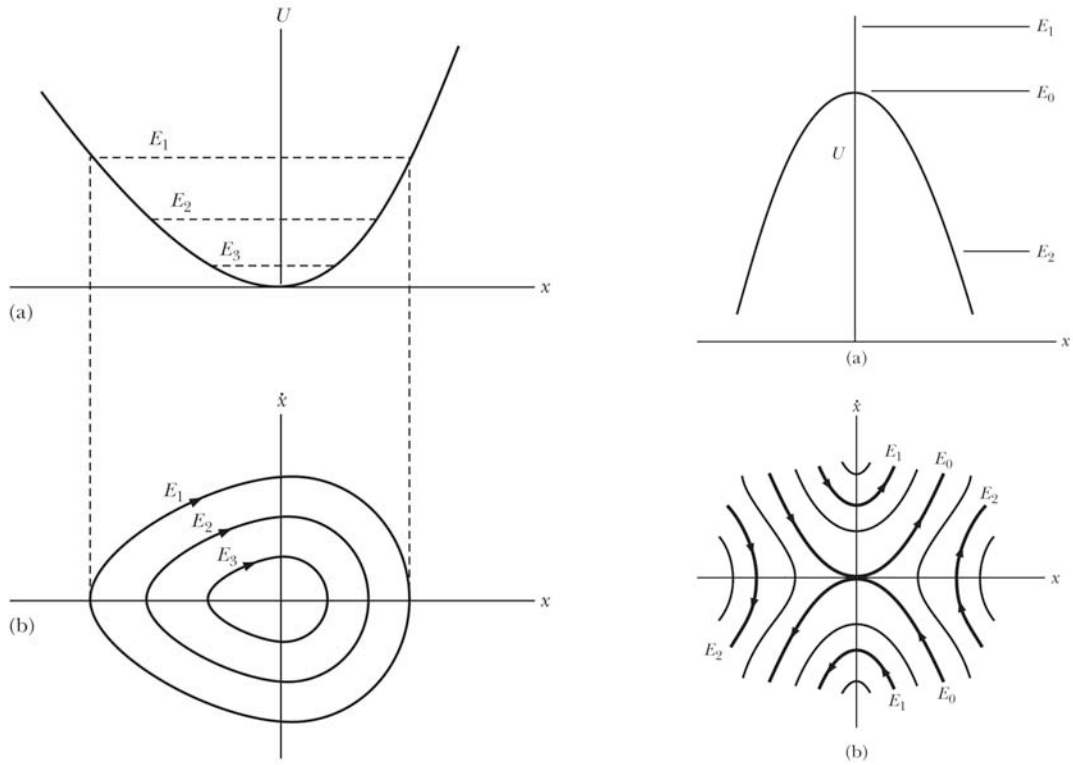


Figure 3. Phase diagrams for stable and unstable equilibrium positions.

<u>The van der Pol Equation</u>

An important type of non-linear differential equation was used to describe the non-linear oscillations in electronic circuits containing vacuum tubes. The van der Pol differential equation has the following form:

$$\ddot{x} + \mu\left(x^2 - a^2\right)\dot{x} + \omega_0{}^2 x = 0$$

This equation is non-linear due to the presence of the x^2 term. One can interpret this equation as the equation of a system that has variable damping. In addition, the "damping" coefficient can be positive and negative, depending on the value of a. Based on what we have seen for damping in Chapter 3, we expect the following behavior for the solution of this differential equation:

- If $|x| >> |a|$, the "damping" coefficient is positive and the amplitude of the motion will decrease.
- If $|x| << |a|$, the "damping" coefficient is negative and the amplitude of the motion will increase.

After a period of time we can imagine that the "damping" coefficient approaches zero and simple harmonic motion, with constant amplitude, will result. The curve mapped out in phase space in this case is called the **limit cycle**, and eventually no matter what the starting point, the system will follow the limit cycle.

To study the time evolution of the van der Pol equation, we need to use numerical methods to solve the differential equation. I will illustrate the use of Mathematica to solve this equation. The following code will allow you to play with the solutions of the van der Pol equation (the code can be found in the file *vanDerPolSolution* in the Mathematica folder under Computing Tools on our website):

```
(* Set the values of the various parameters *)
mu = 0.05;
a = 1;
w0 = 1;

(* Solve the differential equations with the given set of
initial conditions. \
*)
sol = NDSolve[
      {x'[t] == v[t],
        v'[t] == -mu*(x[t]*x[t] - a*a)*v[t] - w0*w0*x[t],
        x[0] == 4,
        v[0] == 0},
      {x, v}, {t, 0, 50*2*Pi}, MaxSteps -> 20000];

(* Plot the solution of the differential equations *)

ParametricPlot[Evaluate[{x[t], v[t]} /. sol], {t, 0, 50*2*Pi},
    PlotRange -> All,
    AxesLabel -> {"x (rad)", "v (rad/s)"}];
```

The results of this type of calculations, for different starting parameters and damping terms, are shown in Figures 4 and 5.

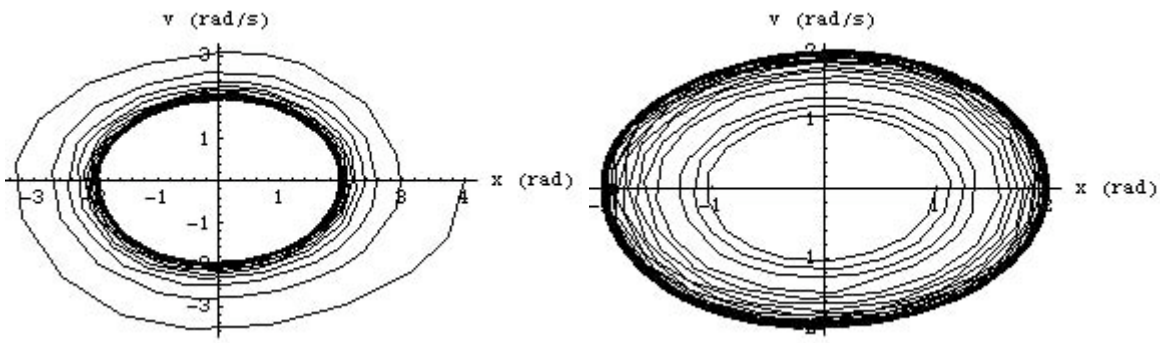


Figure 4. Solution of the van der Pol equation with μ = 0.05, a = 1 rad, ω_0 = 1 rad/s, x_0 = 4 rad, v_0 = 0 rad/s (left) and μ = 0.05, a = 1 rad, ω_0 = 1 rad/s, x_0 = 1 rad, v_0 = 0 rad/s (right).

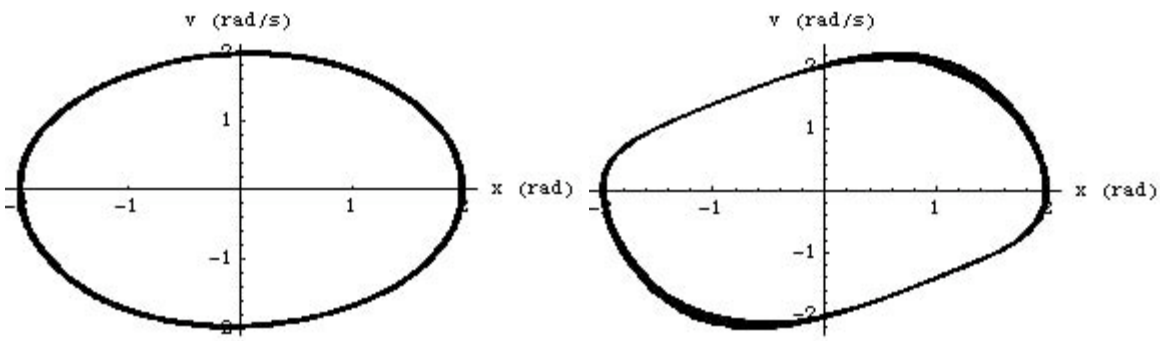


Figure 5. Solution of the van der Pol equation with μ = 0.05, a = 1 rad, ω_0 = 1 rad/s, x_0 = 2 rad, v_0 = 0 rad/s (left) and μ = 0.5, a = 1 rad, ω_0 = 1 rad/s, x_0 = 2 rad, v_0 = 0 rad/s (right).

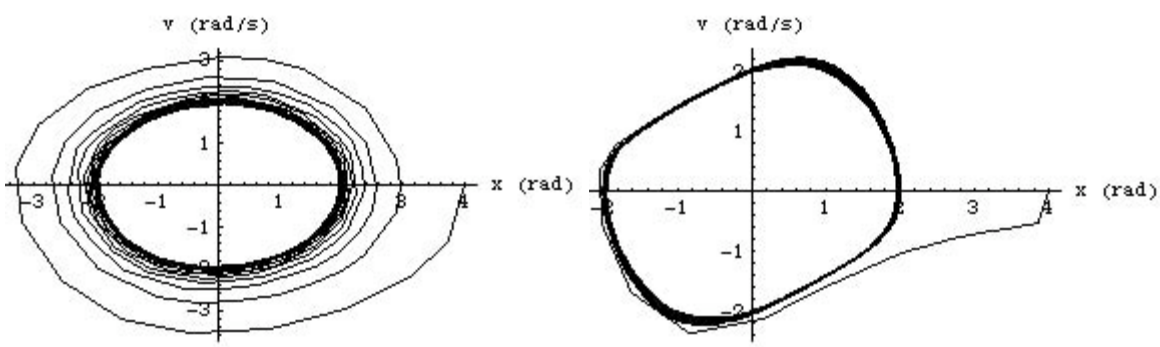


Figure 6. Solution of the van der Pol equation with μ = 0.05, a = 1 rad, ω_0 = 1 rad/s, x_0 = 4 rad, v_0 = 0 rad/s (left) and μ = 0.5, a = 1 rad, ω_0 = 1 rad/s, x_0 = 4 rad, v_0 = 0 rad/s (right).

Looking at Figure 5 we observe that the shape of the limiting curve is a function of the damping parameter. Figure 6 shows that an increase of the damping parameter decreases the time required to approach the limit cycle.

The Plane Pendulum

A good example of a system with non-linear behavior is the plane pendulum. In many introductory physics courses, the simple pendulum is used to introduce harmonic motion. However, the simple pendulum will only exhibit simple harmonic motion if the angular displacement is small (small enough to ensure that $\sin\theta$ is approximately θ). If we can not make the assumption that the angular displacements are small, the simple pendulum will show non-linear behavior. The differential equation describing the system is given by

$$\ddot{\theta} + \omega_0^{\,2} \sin\theta = 0$$

where

$$\omega_0^{\,2} = \frac{g}{l}$$

where l is the length of the pendulum.

Before we look in detail at the phase diagrams for this pendulum, we should first consider what we expect to see. Let's assume that at time t = 0 s, the pendulum is located at its equilibrium position, and it has a linear velocity v_0 at that time.

- If the velocity at $t = 0$ is small, the system will oscillate back and forth between a minimum and maximum value of the angular displacements, and the phase diagram is a closed curve (see Figure 7).
- If the velocity at $t = 0$ is sufficiently large, the system will continue to rotate in one direction, and the angular velocity will show an oscillatory behavior (but without a sign reversal). Examples of this response are shown in Figure 8.
- The limiting case will be the case where the velocity at $t = 0$ is such that the pendulum makes it just to its highest possible position ($y = 2l$). Assuming the mass of the pendulum is connected via a weightless rod to the pivot point, the total energy of the mass will have to be $2mgl$ (kinetic energy will be equal to 0 at the top position). The kinetic energy at the equilibrium must therefore be $2mgl$, and the linear velocity at that postion will be equal to $2\sqrt{gl}$. The angular velocity at the equilibrium position will thus be equal to $(2\sqrt{gl})/l = 2\sqrt{(g/l)} = 2\omega_0$. For the phase diagrams shown in Figures 7 and 8, we have assumed ω_0 = 1 rad/s, and the "critical" angular velocity is thus equal to 2 rad/s.

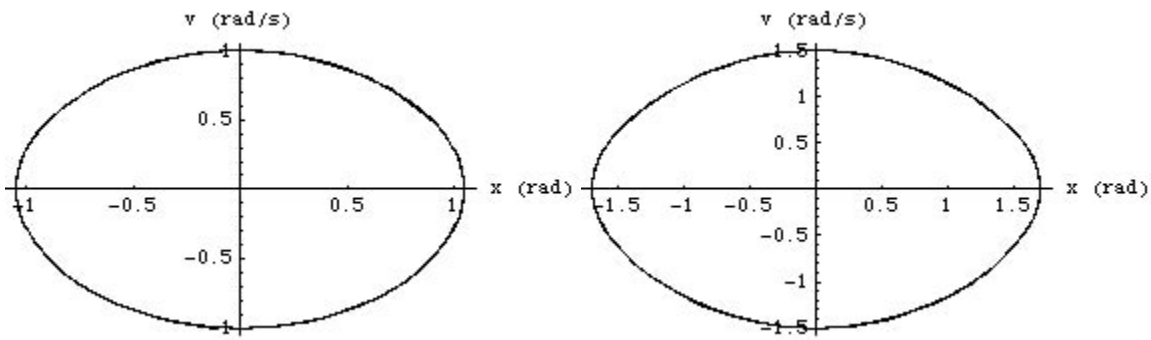


Figure 7. Solution of the differential equation associated with the plane pendulum ω_0 = 1 rad/s, x_0 = 0 rad, v_0 = 1 rad/s (left) and ω_0 = 1 rad/s, x_0 = 0 rad, v_0 = 1.5 rad/s (right).

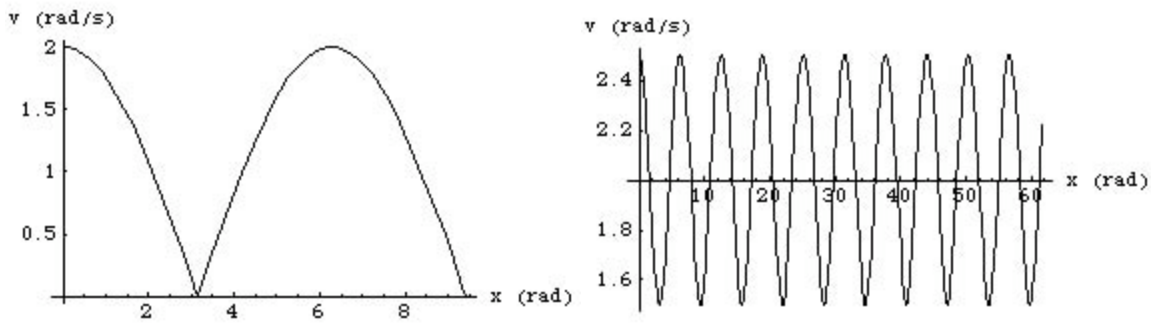


Figure 8. Solution of the differential equation associated with the plane pendulum ω_0 = 1 rad/s, x_0 = 0 rad, v_0 = 2 rad/s (left) and ω_0 = 1 rad/s, x_0 = 0 rad, v_0 = 2.5 rad/s (right).

Figures 7 and 8 were made using Mathematica and can be reproduced using the following code, which is also contained in Mathematica notebook *planePendulum* that is available from our website:

```
(* Set the values of the various parameters *)
w0 = 1;

(* Solve the differential equations with the given set of
initial conditions. \
*)
sol = NDSolve[
      {x'[t] == v[t],
        v'[t] == -w0*w0*Sin[x[t]],
        x[0] == 0,
```

```
        v[0] == 2},
      {x, v}, {t, 0, 50*2*Pi}, MaxSteps -> 20000];

(* Plot the solution of the differential equations *)

ParametricPlot[Evaluate[{x[t], v[t]} /. sol], {t, 0, 5*2*Pi},
    PlotRange -> All,
    AxesLabel -> {"x (rad)", "v (rad/s)"}];
```

Another way to look at the phase diagram is to use contour plots. Rather than solving the differential equation for a specific set of initial parameters, and thus a specific energy, the contour plot provides an overview of the solution for a range of initial conditions. Consider a point in the phase diagram corresponding to an angular position θ and an angular velocity ω. The kinetic and potential energies at this location are:

$$T = \frac{1}{2} I\omega^2 = \frac{1}{2} ml^2\dot{\theta}^2$$

and

$$U = mgl\left(1 - \cos\theta\right)$$

The total energy E is thus equal to

$$E = T + U = \frac{1}{2} ml^2\dot{\theta}^2 + mgl\left(1 - \cos\theta\right)$$

A contour plot, showing the total energy E as function of θ and ω, shows the nature of the motion that will result when we give the pendulum specific initial conditions. An example of a contour plot, obtained for m = 1 kg, l = 1 m is shown in Figure 9. This contour plot was generated using the following Mathematica code, which is also contained in Mathematica notebook *planePendulumContours* that is available from our website:

```
l = 1;
g = 9.8;
ContourPlot[(1/2)*l^2*w^2 + g*l*(1 - Cos[theta]),
    {theta, -4Pi, 4Pi}, {w, -15, 15}];
```

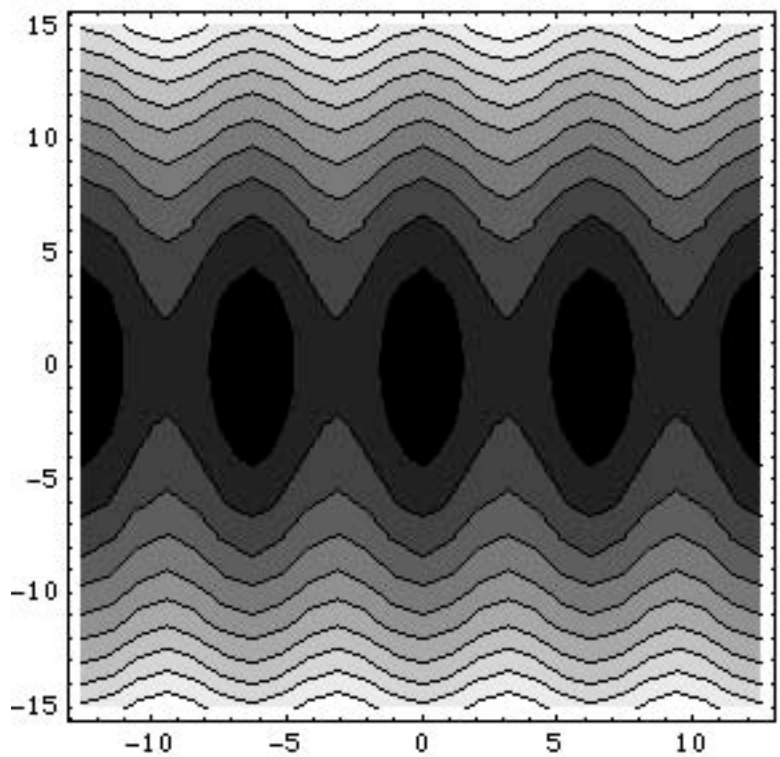


Figure 9. Contour plot of the phase diagram of the plane pendulum (l = 1 m).

Chaos

Important principles of chaos can be studied using a simple driven pendulum. Consider the pendulum shown in Figure 10, which is driven about its pivot point. We assume that the pendulum is damped with a damping coefficient b. The total torque around the pivot point is equal to

$$N = I\ddot{\theta} = ml^2\ddot{\theta}$$

The total torque is equal to the sum of the torques due to the gravitational force, the damping force, and the driving force:

$$N = -b\dot{\theta} - mgl\sin\theta + N_d\cos(\omega t)$$

The differential equation describing the motion of this driven pendulum is given by

$$\ddot{\theta} = -\frac{b}{ml^2}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{N_d}{ml^2}\cos(\omega t)$$

This differential equation is a non-linear equation due to the $\sin\theta$ term. This equation has the following general form:

$$\ddot{x} = -c\dot{x} - \sin x + F\cos(\omega t)$$

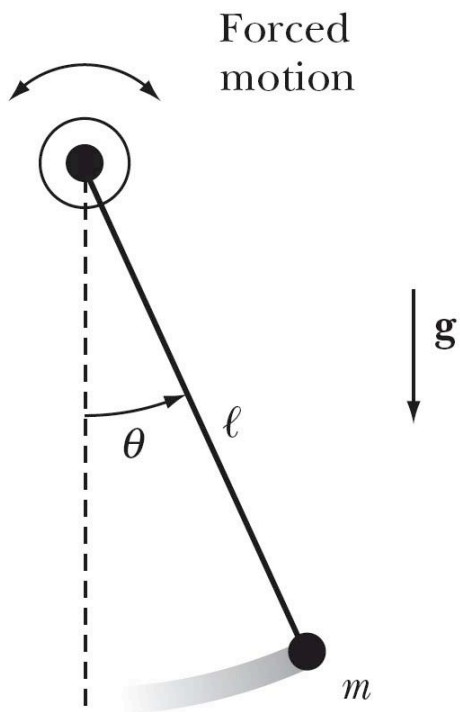


Figure 10. A damped pendulum, driven about its pivot point.

The solution to this equation can be studied using numerical methods, and we will focus on the results obtained with Mathematica. The code required to study the solutions of this equation can be found in the file *chaosPendulum* in the Mathematica folder under Computing Tools on our website:

```
(* Set the values of the various parameters *)
c = 0.2;
w = 0.694;
F = 0.52;
pi = N[Pi];
cycles = 50;
steps = 30;

(* Solve the differential equations with the given set of
initial conditions. \
```

```
*)
sol = NDSolve[
      {x'[t] == v[t],
        v'[t] == -c*v[t] - Sin[x[t]] + F*Cos[w*t],
        x[0] == 0.8,
        v[0] == 0.8},
      {x, v}, {t, 0, (cycles*(2Pi)/w)}, MaxSteps -> 20000];

(* Create a function "reduce" that translate all angles back to
the region \
between - Pi and + Pi *)
reduce[x_] := Mod[x, 2pi] /; Mod[x, 2pi] <= pi;
reduce[x_] := (Mod[x, 2pi] - 2pi) /; Mod[x, 2pi] > pi;

(* Create a table with the angles mapped to the - Pi to Pi
region. *)
points = Flatten[
      Table[
        {x[t], v[t]} /. sol,
        {t, 0, (cycles*(2Pi)/w), (1/steps)(2Pi/w) // N}
        ], 1
      ];
xposition = Table[{i, 1}, {i, Length[points]}];
newpoints = MapAt[reduce, points, xposition];

(* Plot the solution of the differential equations : v[t] vs t.
*)
ParametricPlot[
    Evaluate[{t, v[t]} /. sol], {t, 0, (cycles*(2Pi)/w)},
    PlotRange -> {{((cycles - 20)*(2Pi)/w), (cycles*(2Pi)/w)},
{-3, 3}},
    AxesLabel -> {"t/t0", "v (rad/s)"}
    ];

(* Plot the phase diagram (using the reduced angle). *)
ListPlot[
    newpoints,
    PlotRange -> {{-3, 3}, {-3, 3}},
    AxesLabel -> {"x (rad)", "v (rad/s)"}
    ];
```

```
(* Make the poincare plot. *)
poincare = Table[newpoints[[n]], {n, 1 + 25steps,
Length[points], steps}];
Length[poincare];
ListPlot[
    poincare,
    PlotRange -> {{-3, 3}, {-3, 3}},
    AxesLabel -> {"x (rad)", "v (rad/s)"},
    PlotStyle -> PointSize[0.015]
    ];
```

There are many different ways of looking at the results of this calculation. Consider the case where $c = 0.2$, $\omega = 0.694$, and $F = 0.52$. A graph showing the velocity versus time is shown in Figure 11.

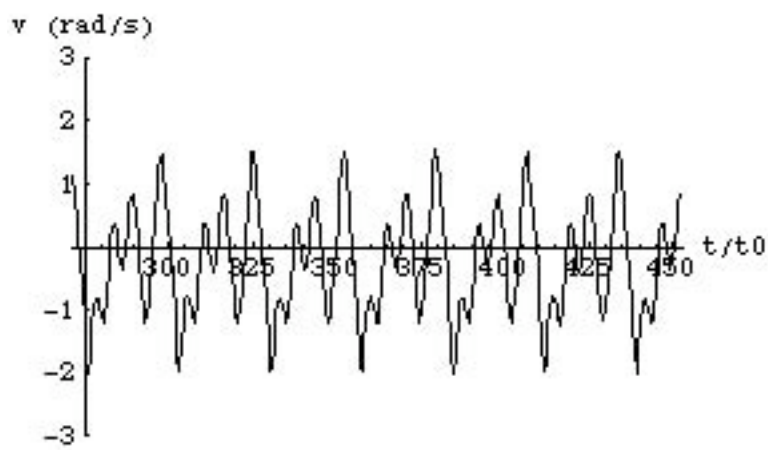
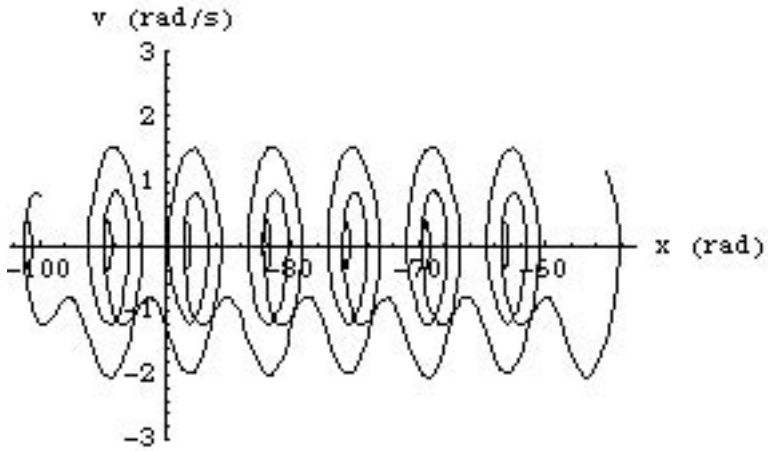


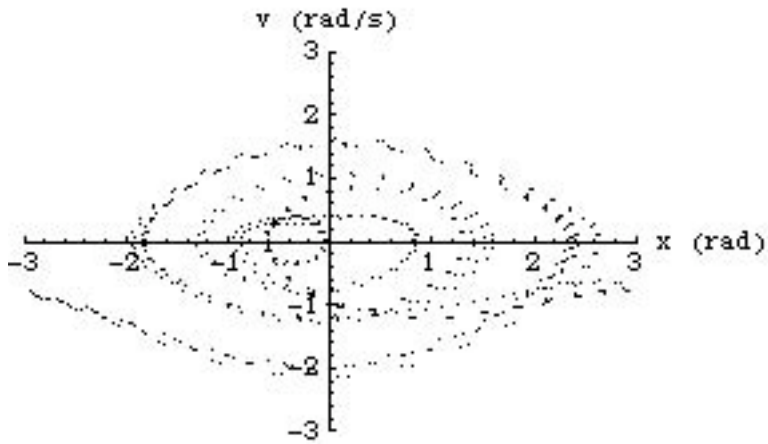
Figure 11. Velocity versus time for the driven pendulum with c = 0.2, ω = 0.694, and F = 0.52.

The period shown in Figure 11 does not include the initial period when transient effects may dominate. The motion is periodic, but not simple harmonic. The average velocity is negative, indicating that on average the solution will move to more negative angles.

Another way to look at the solution is to construct the phase diagram. The phase diagram for the solution shown in Figure 11 is shown in Figure 12. Figure 12a shows the velocity versus angle. As expected, the system will move towards more negative angles. Figure 12b shows the similar information to what is shown in Figure 12a, except that the angle is reduced to an angle between $-\pi$ and $+\pi$.



a)



b)

Figure 12. Phase diagrams for the driven pendulum with c = 0.2, ω = 0.694, and F = 0.52. The diagram shown in b) contains the same information as in a) except that the angles are mapped on the range between –π and +π.

The phase diagrams shown in Figure 12 are often difficult to interpret. An approach that was developed to emphasize the possible periodic nature of the solution is the Poincare representation. In this representation, we plot the velocity versus position at times separated by one period ($2\pi/\omega$). If the motion is periodic, each new point in the Poincare diagram will be plotted at the same position as the previous point: the plot shows thus a single point. **Note: the number of points shown will provide information about the periodicity; the actual position does not provide much information.** The Poincare representation of the phase diagrams shown in Figure 12 is shown in Figure 13. There are 3 points in the Poincare representation, suggesting a periodic solution (but not simple harmonic).

The solution of the driven pendulum is very sensitive to the parameters that are being used in the calculation. The transition of chaos is characterized by an increase in the number of points in the Poincare diagrams. Consider what happens if we change one of the parameters of our differential equation. Figure 14 shows the results of a calculation when we increase F from 0.52

to 0.62. The solution has become a simple harmonic, as can be seen by the fact that the Poincare plot contains only a single point.

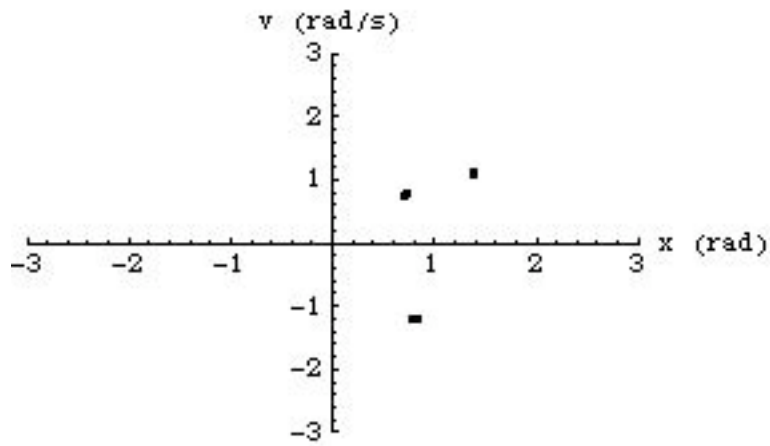


Figure 13. Poincare representation of the motion of a driven pendulum with c = 0.2, ω = 0.694, and F = 0.52.

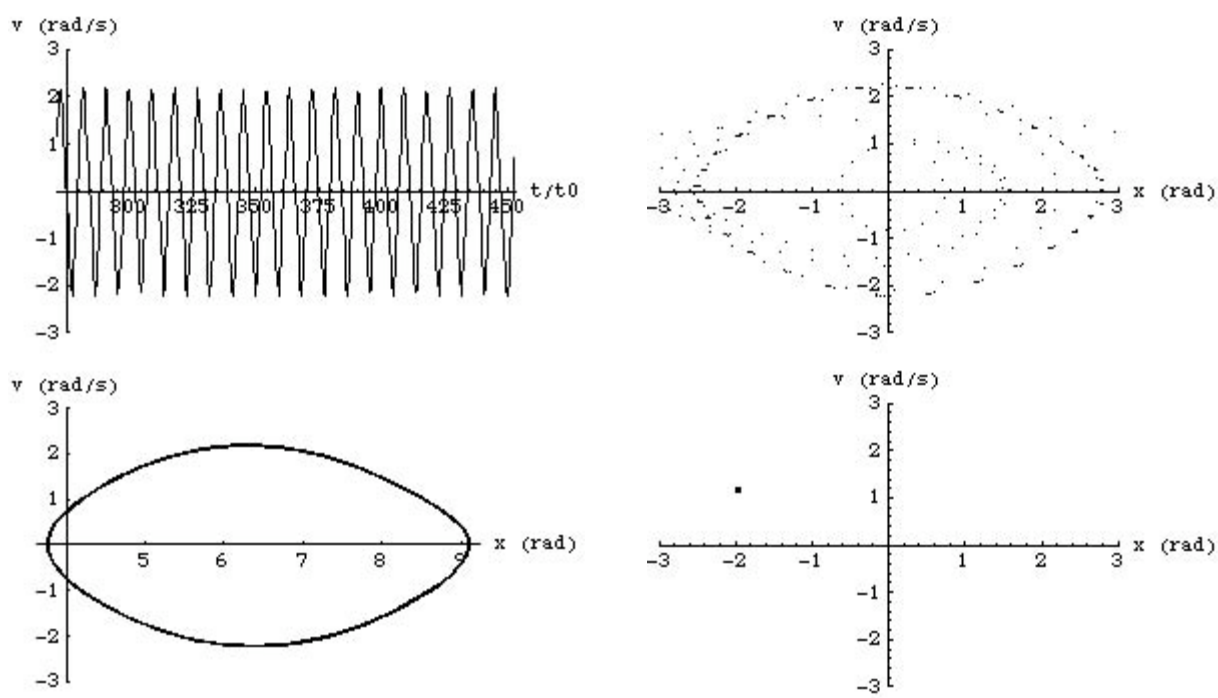


Figure 13. Solutions of the motion of a driven pendulum with c = 0.2, ω = 0.694, and F = 0.62.

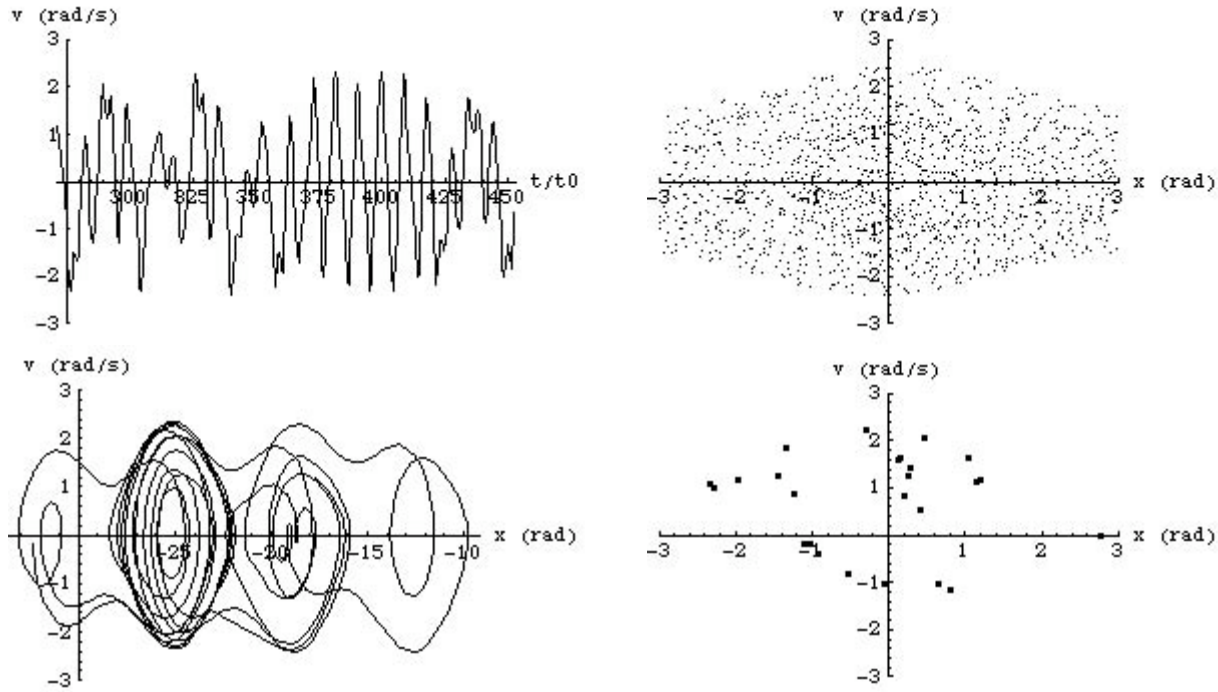


Figure 14. Solutions of the motion of a driven pendulum with c = 0.2, ω = 0.694, and F = 0.72.

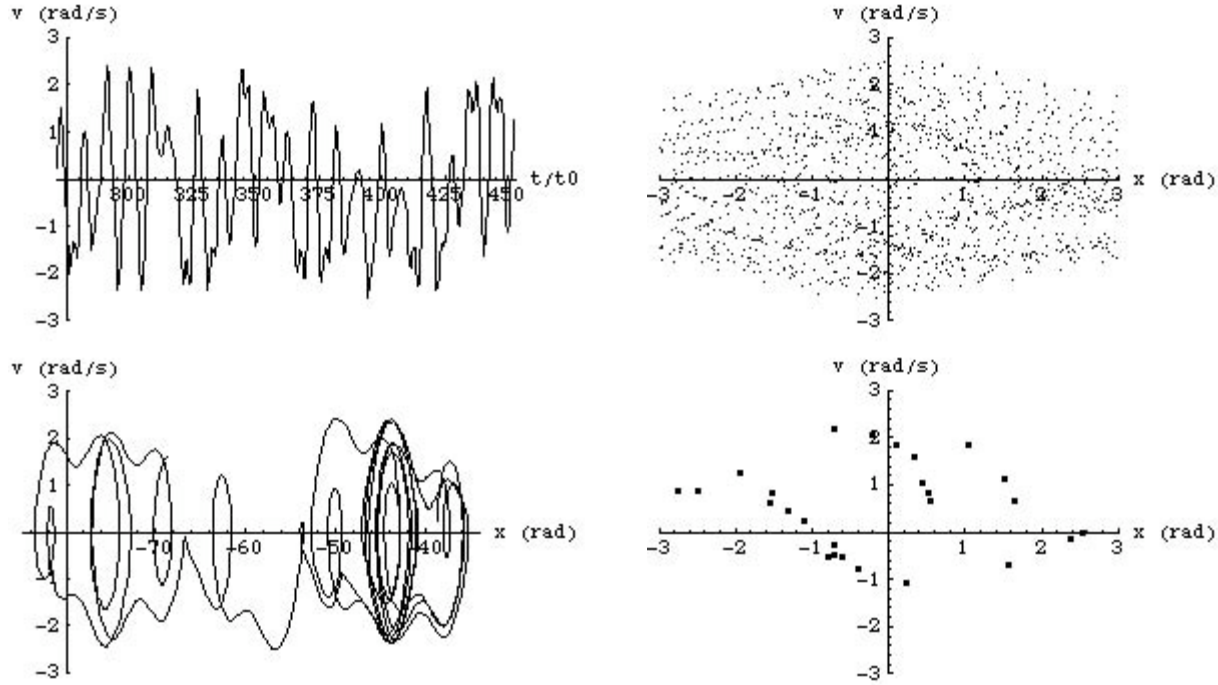


Figure 15. Solutions of the motion of a driven pendulum with c = 0.2, ω = 0.694, and F = 0.82.

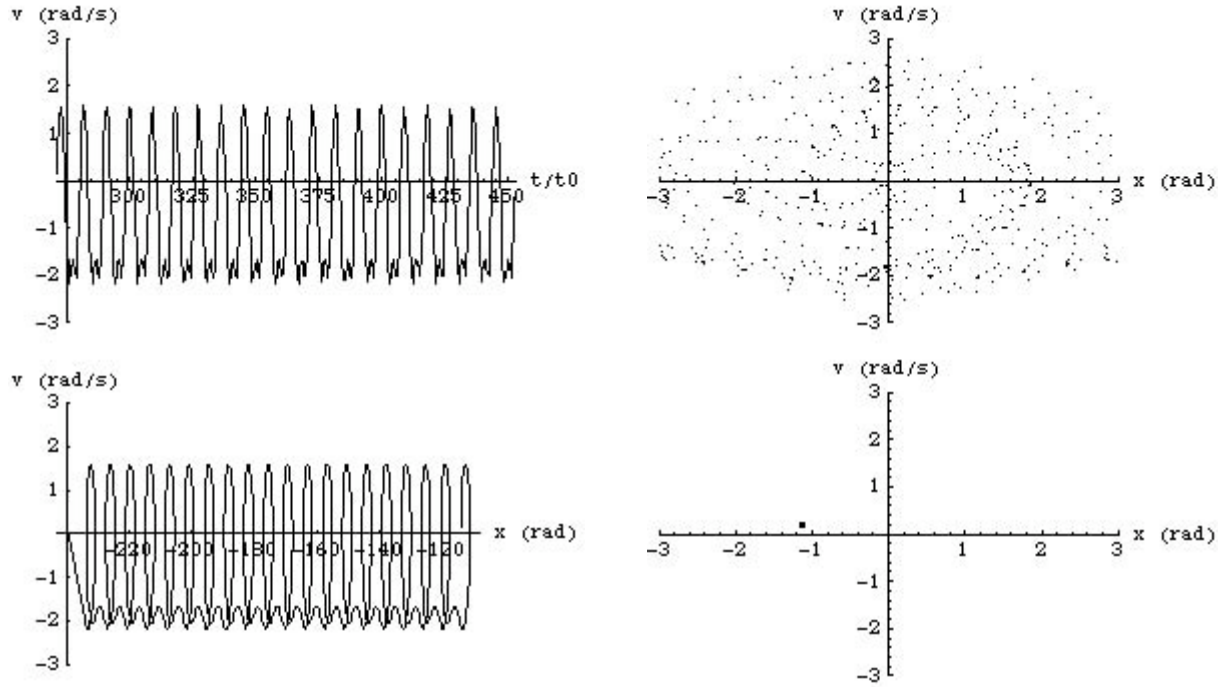


Figure 15. Solutions of the motion of a driven pendulum with c = 0.2, ω = 0.694, and F = 0.92.

Figures 14, 15, and 16 show the results of increasing the force F to 0.72, 0.82, and 0.92. The solution is chaotic when $F = 0.72$ and $F = 0.82$, but the solution is periodic when $F = 0.92$.

Logistic Equations

Chaos can also develop when we work with maps. Maps are used to describe the evolution of a system, and can developed if we know how the future depends on the immediate past. For example, consider a situation where the value of a system parameter at time t_{N+1}, x_{N+1}, depends on the value of this parameter at time t_N, x_N:

$$x_{N+1} = \alpha x_n \left(1 - x_n\right)$$

The function describing this relation is called the **logistic equation**, and it depends on both x and on α:

$$f\left(\alpha, x\right) = \alpha x\left(1 - x\right)$$

For certain values of α, the system will evolve towards a stable equilibrium while for other starting value the system may evolve towards more than one possible solution. Two examples, for two different values of α, are shown in Figure 16.

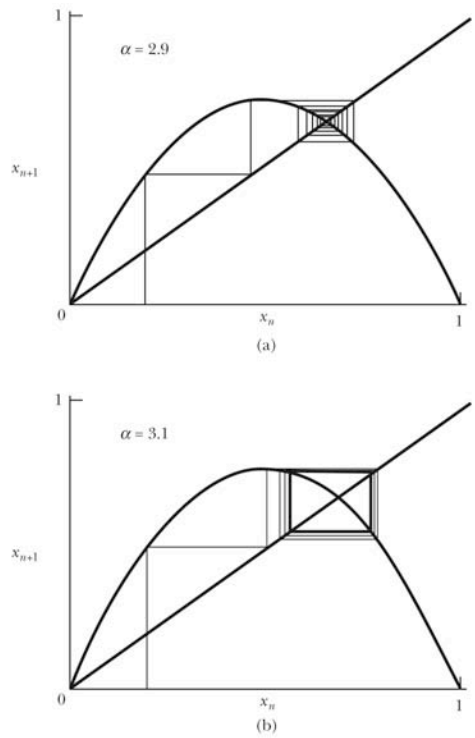


Figure 16. Examples of the logistic equation map for two different values of α.

In the example shown in Figure 16b, there are two solutions to the logistic map. For higher value of α there may be more solutions. The best way to view the possible modes of evolution is to use a bifurcation diagram. In a bifurcation diagram, we plot the value of x_n as function of the parameter α. We can construct a bifurcation diagram using numerical methods, and we will focus on the results obtained with Mathematica. The code required to study the solutions of this equation can be found in the file *bifurcation* in the Mathematica folder under Computing Tools on our website. A bifurcation diagram for the logistic equation discussed in this section is shown in Figure 17. The two diagrams in Figure 17 show the bifurcation diagrams that are created for a starting value $x_0 = 0.1$ (left) and $x_0 = 0.9$ (right). We see that for value of α below 3 there is only one solution. For values of α between 3 and 3.45, there are two possible solutions. For values between 3.45 and 3.55 there are four possible solutions. Above $\alpha = 3.55$ the system has many possible solutions, and the system is said to be chaotic. However, as Figure 17 shows, for very specific values of α the system returns back to a highly-ordered state with a small number of solutions. As can be seen, the bifurcation diagram does not change when we change the initial conditions, except for the special cases shown in Figure 18 where $x_0 = 0.0$ (left) and $x_0 = 1.0$ (right).

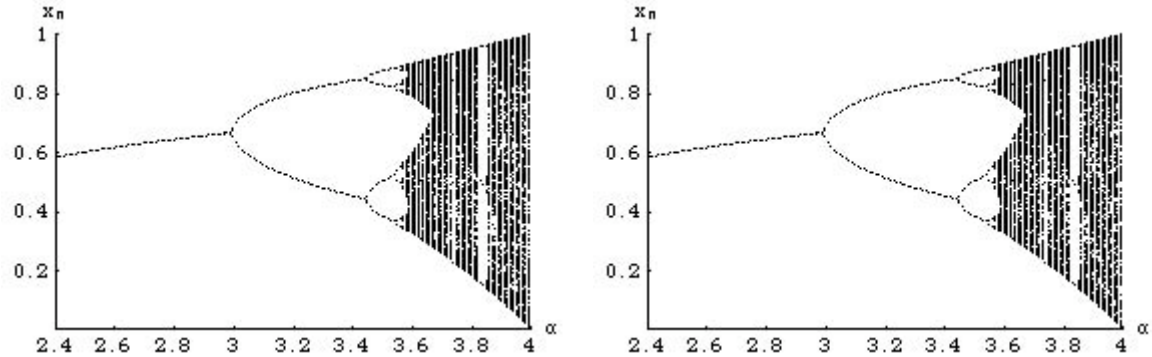


Figure 17. Bifurcation diagrams for $f(\alpha, x) = \alpha\, x(1 - x)$ for a starting value of $x_0 = 0.1$ (left) and $x_0 = 0.9$ (right).

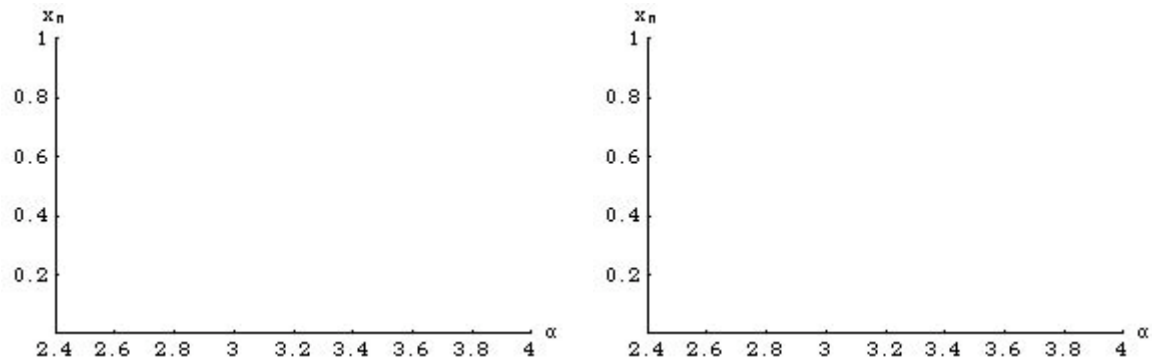


Figure 18. Bifurcation diagrams for $f(\alpha, x) = \alpha\, x(1 - x)$ for a starting value of $x_0 = 0.0$ (left) and $x_0 = 1.0$ (right).

Changes to the logistic equation will change the bifurcation diagram. Consider for example the following logistic equation:

$$f(\alpha, x) = \alpha x (1 - x)^2$$

The corresponding bifurcation diagram is shown in Figure 19. Many of the features are similar, except that we see that the range the first doubling of the solutions does not occur until α exceeds 4.

When the time evolution of a system depends sensitively on the initial conditions, the system is said to show the characteristics of chaos. In the bifurcation diagrams shown in Figure 17, we see that a particular system can have non-chaotic behavior for a specific range of parameters and chaotic behavior for a different range of parameters. A good way to examine the development of chaos is by looking at the Lyapunov exponent λ. The Lyapunov exponent is based on the differences between solutions if we make a small change in the initial conditions. Consider two

starting values of x: $x_0 + \varepsilon$ and x_0. The difference between the solutions for these starting conditions is

$$d^n = f^n(\alpha, x_0 + \varepsilon) - f^n(\alpha, x_0)$$

It turns out that this difference grows as

$$d^n = \varepsilon e^{n\lambda}$$

Combining these equations we see that

$$\lambda = \frac{1}{n}\ln\left(e^{n\lambda}\right) = \frac{1}{n}\ln\left[\frac{f^n(\alpha, x_0 + \varepsilon) - f^n(\alpha, x_0)}{\varepsilon}\right] = \frac{1}{n}\ln\left[\left.\frac{df^n(\alpha, x)}{dx}\right|_{x_0}\right]$$

Using the chain rule for derivations, we can rewrite this equation as

$$\lambda = \frac{1}{n}\ln\left[\left.\frac{df^n(\alpha, x)}{dx}\right|_{x_0}\right] = \frac{1}{n}\ln\left[\left.\frac{df(\alpha, x)}{dx}\right|_{x_{n-1}}\left.\frac{df(\alpha, x)}{dx}\right|_{xx_{n-2}} \cdots \left.\frac{df(\alpha, x)}{dx}\right|_{x_1}\left.\frac{df(\alpha, x)}{dx}\right|_{x_0}\right]$$

If we take the limit of n to infinity, we get the definition of the Lyapunov exponent

$$\lambda = \lim_{x\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}\ln\left|\frac{df(\alpha, x_i)}{dx}\right|$$

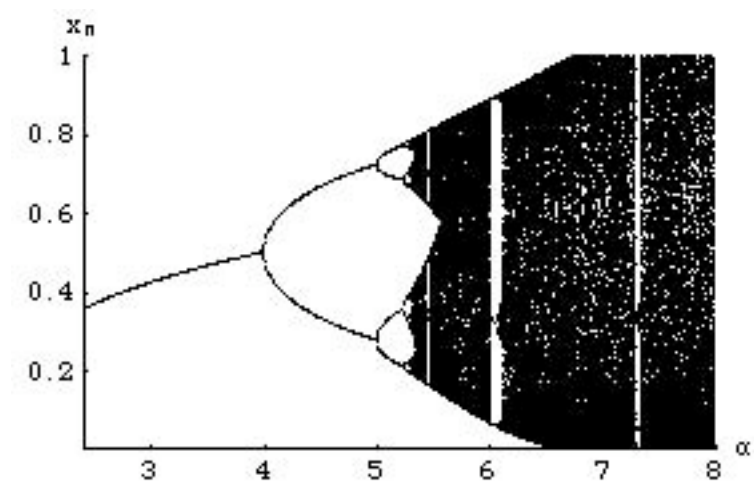


Figure 19. Bifurcation diagram for $f(\alpha, x) = \alpha x(1 - x)^2$ for a starting value of $x_0 = 0.1$.

A plot of the Lyapunov exponent as function of α for the logistic function $f(\alpha, x) = \alpha\, x(1 - x)$ is shown in Figure 20.

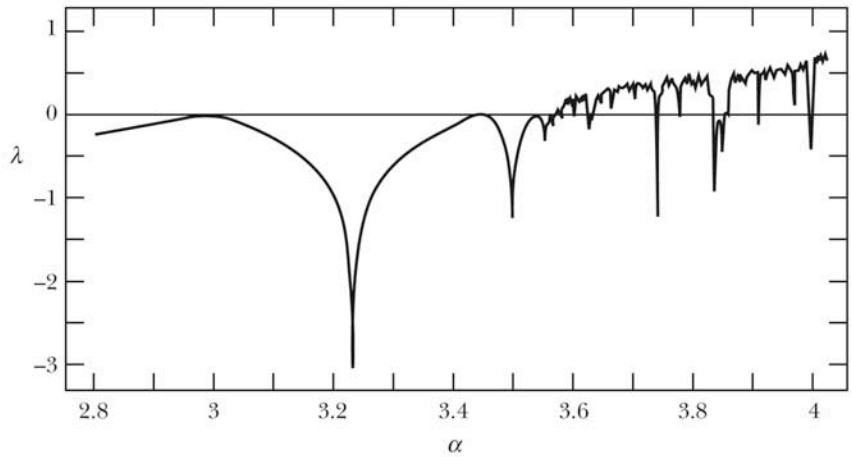


Figure 20. Lyapunov exponent as function of α for the logistic equation $f(\alpha, x) = \alpha\, x(1 - x)$.

If the Lyapunov exponent is negative, the solutions will eventually converge. In Figure 20 we can thus easily identify the region where a stable solution is achieved. Bifurcation (doubling of the number of solutions) will occur when the Lyapunov exponent is zero. Chaos will occur when the Lyapunov exponent is larger than 0.