
Classical Mechanics

Phy 235, Lecture 11.

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Hamilton's Principle

" Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies. "

$$\delta \int_{t_1}^{t_2} (T - U) dt = 0$$

The quantity $T - U$ is called the **Lagrangian L** .

The Lagrange equation(s) of motion are:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0$$

Problem 7.12

A particle of mass m rests on a smooth plane.

The plane is raised to an inclination angle θ at a constant rate a ($\theta = 0^\circ$ at $t = 0$ s), causing the particle to move down the plane.

Determine the motion of the particle.

Generalized coordinates.

- So far we have expressed the Lagrangian in terms of (generalized) position and (generalized) velocities:

$$L = T - U = \frac{1}{2} m \sum_{i=1}^3 \dot{x}_i^2 - U(x_i)$$

- An alternative is to express the Lagrangian in terms of (generalized) position and (generalized) momenta. For example:

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{\partial T}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Conservation Laws – Part I.

- Conservation of energy:
 - Lagrangian does not depend on time explicitly.
 - If L does not depend explicitly on time, it can be shown that

$$L - \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) = \text{constant} = -H$$

- The constant H is called the Hamiltonian of the system:

$$H = \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) - L$$

Problem 7.22.

A particle of mass m moves in one dimension under the influence of a force F :

$$F(x, t) = \frac{k}{x^2} e^{-(t/\tau)}$$

where k and τ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy and discuss the conservation of energy for the system.



2 Minute 2 Second Intermission.

- Since paying attention for 1 hour and 15 minutes is hard when the topic is physics, let's take a 2 minute 2 second intermission.
- You can:
 - Stretch out.
 - Talk to your neighbors.
 - Ask me a quick question.
 - Enjoy the fantastic music.



Conservation Laws – Part II.

- Conservation of linear momentum:
 - Lagrangian should not be effected by a translation of space.
- Conservation of angular momentum:
 - Lagrangian should not be effected by a rotation of space.

Canonical Equations of Motion

Lagrange equations of motion in terms of generalized momentum:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i = \frac{\partial L}{\partial q_i}$$

The Hamiltonian H can be written in terms of the generalized momenta as

$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \sum_j \dot{q}_j p_j - L$$

Hamilton's Equations of Motion

- For each coordinate: **two** equations of motion.
 - For each coordinate there is only one Lagrange equation of motion.
- Equations of motion are **first order differential equations**.
 - The Lagrange equations of motion are second order differential equations.

$$\frac{\partial H}{\partial p_j} - \dot{q}_j = 0$$

$$\frac{\partial H}{\partial q_j} + \dot{p}_j = 0$$

$$\frac{\partial H}{\partial t} + \frac{\partial L}{\partial t} = 0$$

Problem 7.38

The potential for an anharmonic oscillator is $U = kx^2/2 + bx^4/4$ where k and b are constants.

Find Hamilton's equations of motion.

Problem 7.28

The force F that is provided fixed the potential U :

$$U = -\frac{k}{r}$$

Use plane polar coordinates and find Hamilton's equations of motion.

Problem 7.24

Consider a simple plane pendulum consisting of a mass m attached to a string of length l . After the pendulum is set into motion, the length of the string is shortened at a constant rate:

$$\frac{dl}{dt} = -\alpha = \text{constant}$$

The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

End of Chapter 7

- We now have finished the material that will be covered on midterm exam# 2: chapters 5 – 7.
- Midterm exam # 2 will take place on Tuesday October 23 between 8 am and 9.30 am in B&L 109.

ENOUGH FOR TODAY?