Classical Mechanics
Phy 235, Lecture 23.

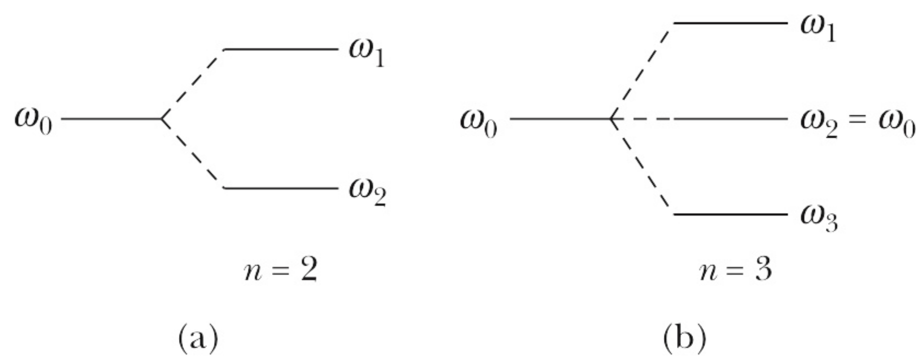
Frank L. H. Wolfs
Department of Physics and Astronomy
University of Rochester

The General Problem of Coupled Oscillations.

Observations from our studies of 2 coupled oscillators:

- The coupling in a system with two degrees of freedom results in two characteristic frequencies.
- The two characteristic frequencies in a system with two degree of freedom are pushed towards lower and higher energies compared to the non-coupled frequency.

Today we will expand the discussion to *n* coupled oscillators.



N Coupled Oscillators

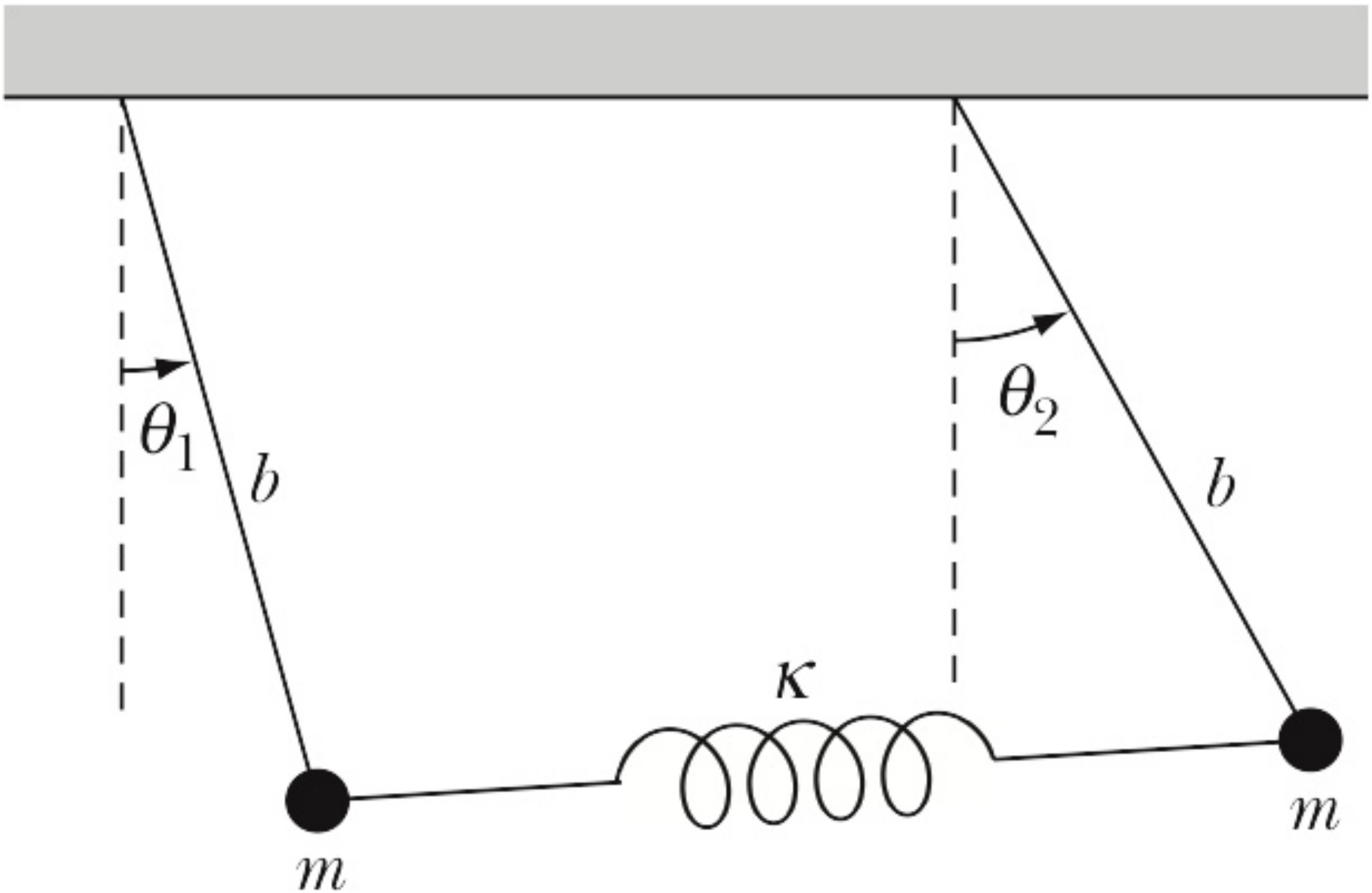
- We will have n coupled equations (A and m are the amplitude and mass tensors):

$$\sum_k \left(A_{kj} - \omega^2 m_{kj} \right) a_k = 0$$

- This set of equation will have non-trivial solutions if

$$\begin{vmatrix} A_{11} - \omega^2 m_{11} & A_{12} - \omega^2 m_{12} & A_{13} - \omega^2 m_{13} & \ldots\ldots \\ A_{12} - \omega^2 m_{12} & A_{22} - \omega^2 m_{22} & A_{23} - \omega^2 m_{23} & \ldots\ldots \\ A_{13} - \omega^2 m_{13} & A_{32} - \omega^2 m_{32} & A_{33} - \omega^2 m_{33} & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} = 0$$

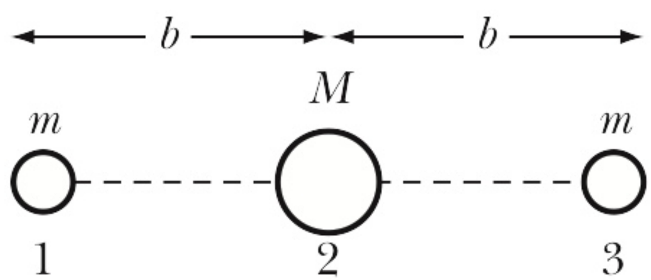
Example 12.4.



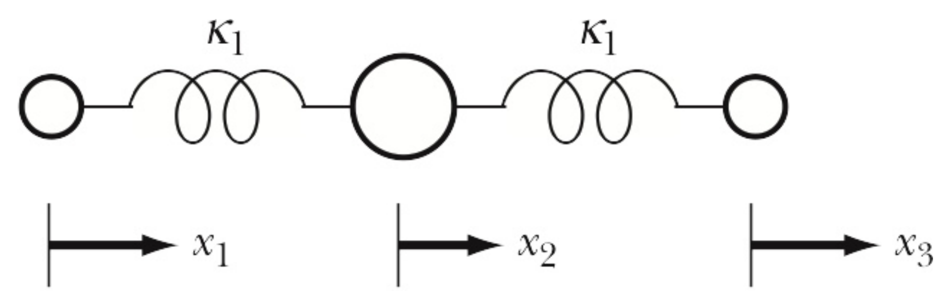
Steps

- Follow these steps in order to solve most coupled oscillator problems:
  - Choose generalized coordinates.
  - Determine the A and m tensors.
  - Determine the eigen frequency and the eigen vectors.
  - Determine the scale factors required to match the initial conditions.
  - Determine the normal coordinates.

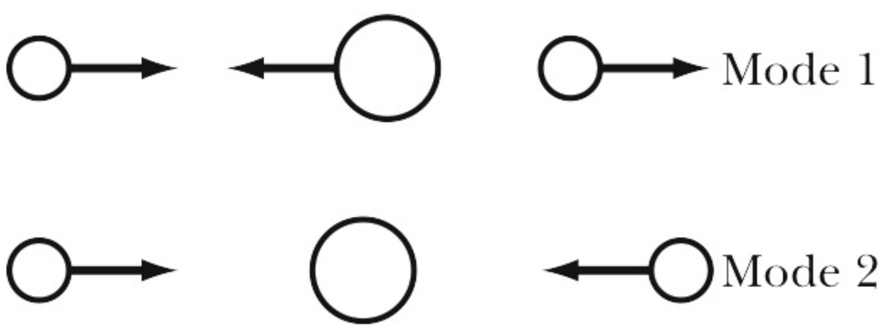
Molecular Vibrations



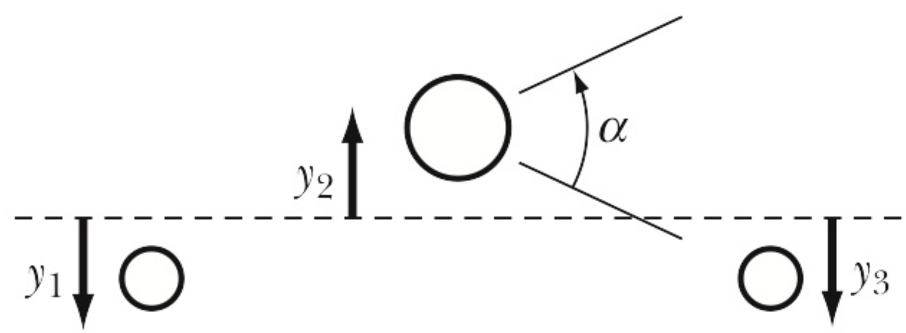
(a) Linear triatomic molecule



(b) Longitudinal description



(c) Longitudinal normal modes



(d) Transverse normal mode

2 Minute 41 Second Intermission.

- Since paying attention for 1 hour and 15 minutes is hard when the topic is physics, let's take a 2 minute 41 second intermission.
- You can:
  - Stretch out.
  - Talk to your neighbors.
  - Ask me a quick question.
  - Enjoy the fantastic music.

Problem 12.21.

Three oscillators of equal mass m are coupled such that the potential energy of the system is given by

$$U = \frac{1}{2}\left[\kappa_1\left(x_1^2 + x_3^2\right) + \kappa_2 x_2^2 + \kappa_3\left(x_1 x_2 + x_2 x_3\right)\right]$$

where

$$\kappa_3 = \sqrt{2\kappa_1\kappa_2}$$

Find the eigen frequencies by solving the secular equation. What is the physical interpretation of the zero-frequency mode?

General steps to solve this type of problems.

- Find {A}.

$$\{\mathbf{A}\} = \begin{bmatrix} \kappa_1 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 \end{bmatrix}$$

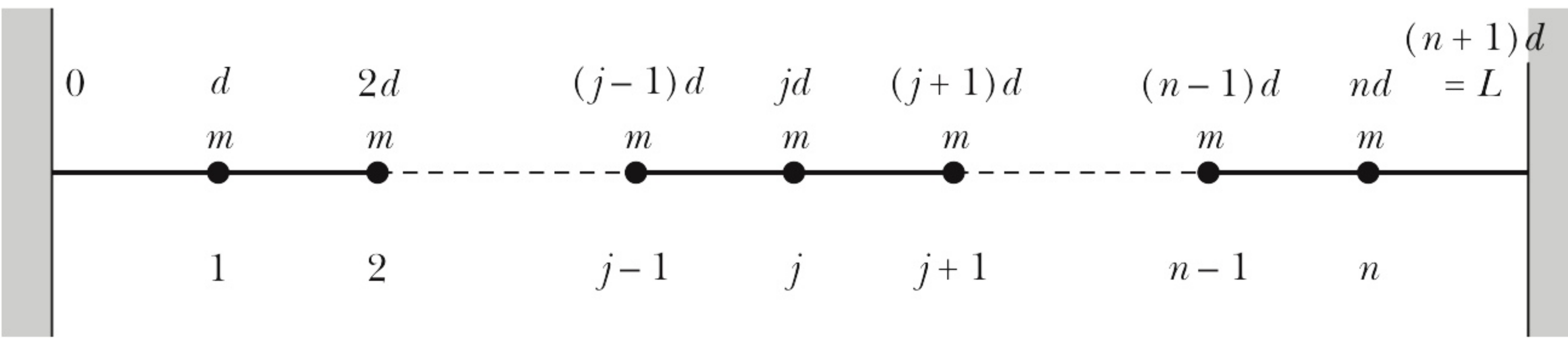
- Find {m}.

$$\{\mathbf{m}\} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

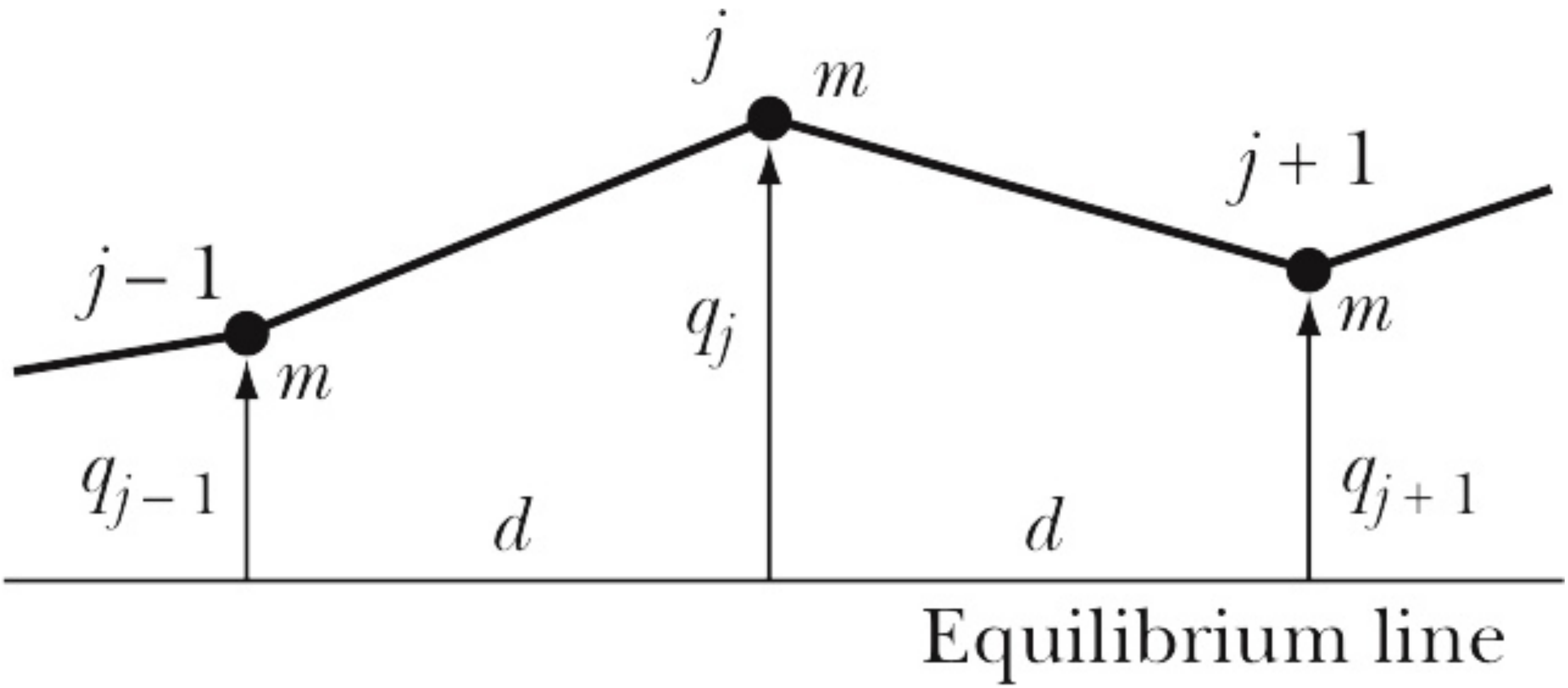
- Solve secular determinant.

$$\begin{vmatrix} \kappa_1 - m\omega^2 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 - m\omega^2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 - m\omega^2 \end{vmatrix} = 0$$

The loaded string.



The loaded string.
Motion in the vertical direction.



The loaded string.

- General solution:

$$q_j(t) = \sum_s \beta_s \sin\left(j \frac{s\pi}{(n+1)} \right) e^{i\omega_s t}$$

- The frequency is given by

$$\omega_s = 2\sqrt{\frac{\tau}{md}} \sin\left(\frac{s\pi}{2(n+1)} \right)$$

Preview Chapter 13: Waves.

- Consider the following:
  - Increase the number of masses n to infinity.
  - Decrease the distance d to zero such that $(n + 1)d = L$.
  - Decrease the mass m to zero such that m/d = constant.
- In this case, the displacement q can be written as:

$$q_j(t) = \sum_s \beta_s \sin\left(s\pi \frac{x}{L}\right) e^{i\omega_s t} = q(x,t)$$

Wave solution.

- If the displacement and velocity at $t = 0$ are known, the constants in the expression for q can be determined.
- In order to find these constants, we multiply each side by $\sin(r\pi x/L)$ and integrate x between 0 and L. We use the following fact:

$$\int_0^L \sin\left(s\pi\frac{x}{L}\right)\sin\left(r\pi\frac{x}{L}\right)dx =$$

$$= \frac{1}{2}\int_0^L \left\{\cos\left(\left(s-r\right)\pi\frac{x}{L}\right) - \cos\left(\left(s+r\right)\pi\frac{x}{L}\right)\right\}dx = \frac{L}{2}\delta_{rs}$$

Energy in a string.

- Kinetic energy:

$$T = \int_0^L dT = \frac{L}{4}\rho\sum_s\left(\omega_s v_s\cos\left(\omega_s t\right) + \omega_s\mu_s\sin\left(\omega_s t\right)\right)^2$$

- Potential energy:

$$U = \frac{\tau}{2}\int_0^L\left(\frac{\partial q}{\partial x}\right)^2 dx = \frac{\rho L}{4}\sum_s\omega_s^{\ 2}\left(\mu_s\cos\left(\omega_s t\right) - v_s\sin\left(\omega_s t\right)\right)^2$$

- Total energy:

$$E = T + U = \frac{\rho L}{4}\sum_s\left\{\omega_s^{\ 2}\left(\mu_s^{\ 2} + v_s^{\ 2}\right)\right\} = \text{constant}$$

ENOUGH FOR TODAY?