

Classical Mechanics Phy 235, Lecture 23.

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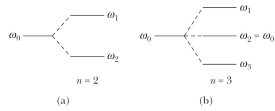
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The General Problem of Coupled Oscillations.

Observations from our studies of 2 coupled oscillators:

- The coupling in a system with two degrees of freedom results in two characteristic frequencies.
- The two characteristic frequencies in a system with two degree of freedom are pushed towards lower and higher energies compared to the non-coupled frequency.



Today we will expand the discussion to n coupled oscillators.

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N Coupled Oscillators

- We will have n coupled equations (A and m are the amplitude and mass tensors):

$$\sum_k (A_{kj} - \omega^2 m_{kj}) a_k = 0$$

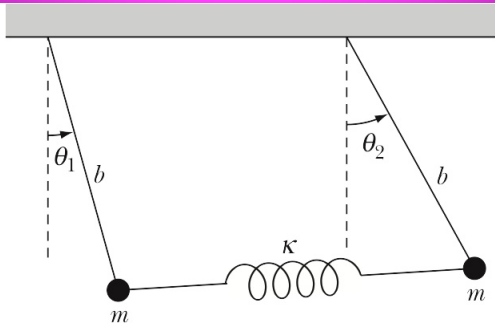
- This set of equation will have non-trivial solutions if

$$\begin{vmatrix} A_{11} - \omega^2 m_{11} & A_{12} - \omega^2 m_{12} & A_{13} - \omega^2 m_{13} & \dots & \dots \\ A_{12} - \omega^2 m_{12} & A_{22} - \omega^2 m_{22} & A_{23} - \omega^2 m_{23} & \dots & \dots \\ A_{13} - \omega^2 m_{13} & A_{32} - \omega^2 m_{32} & A_{33} - \omega^2 m_{33} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0$$

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Example 12.4.



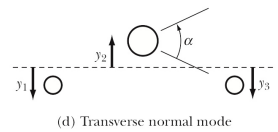
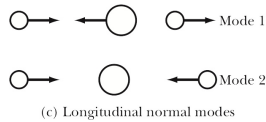
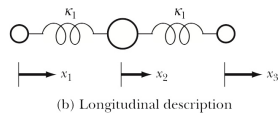
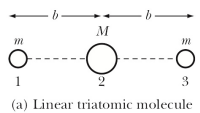
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Steps

- Follow these steps in order to solve most coupled oscillator problems:
 - Choose generalized coordinates.
 - Determine the A and m tensors.
 - Determine the eigen frequency and the eigen vectors.
 - Determine the scale factors required to match the initial conditions.
 - Determine the normal coordinates.

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Molecular Vibrations



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2 Minute 41 Second Intermission.

- Since paying attention for 1 hour and 15 minutes is hard when the topic is physics, let's take a 2 minute 41 second intermission.
- You can:
 - Stretch out.
 - Talk to your neighbors.
 - Ask me a quick question.
 - Enjoy the fantastic music.



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Problem 12.21.

Three oscillators of equal mass m are coupled such that the potential energy of the system is given by

$$U = \frac{1}{2} [\kappa_1 (x_1^2 + x_3^2) + \kappa_2 x_2^2 + \kappa_3 (x_1 x_2 + x_2 x_3)]$$

where

$$\kappa_3 = \sqrt{2\kappa_1\kappa_2}$$

Find the eigen frequencies by solving the secular equation. What is the physical interpretation of the zero-frequency mode?

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General steps to solve this type of problems.

- Find $\{\mathbf{A}\}$.

$$\{\mathbf{A}\} = \begin{bmatrix} \kappa_1 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 \end{bmatrix}$$

- Solve secular determinant.

$$\begin{vmatrix} \kappa_1 - m\omega^2 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 - m\omega^2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 - m\omega^2 \end{vmatrix} = 0$$

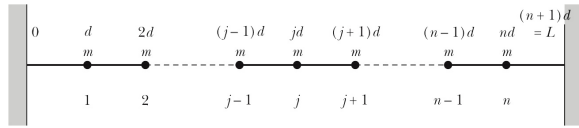
- Find $\{\mathbf{m}\}$.

$$\{\mathbf{m}\} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

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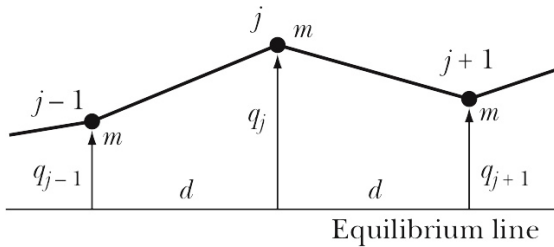
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The loaded string.



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The loaded string. Motion in the vertical direction.



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The loaded string.

- General solution:

$$q_j(t) = \sum_s \beta_s \sin\left(j \frac{s\pi}{(n+1)}\right) e^{i\omega_s t}$$

- The frequency is given by

$$\omega_s = 2\sqrt{\frac{\tau}{md}} \sin\left(\frac{s\pi}{2(n+1)}\right)$$

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Preview Chapter 13: Waves.

- Consider the following:

- Increase the number of masses n to infinity.
- Decrease the distance d to zero such that $(n + 1)d = L$.
- Decrease the mass m to zero such that $m/d = \text{constant}$.

- In this case, the displacement q can be written as:

$$q_j(t) = \sum_s \beta_s \sin\left(s\pi \frac{x}{L}\right) e^{i\omega_s t} = q(x, t)$$

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Wave solution.

- If the displacement and velocity at $t = 0$ are known, the constants in the expression for q can be determined.

- In order to find these constants, we multiply each side by $\sin(r\pi x/L)$ and integrate x between 0 and L . We use the following fact:

$$\begin{aligned} \int_0^L \sin\left(s\pi \frac{x}{L}\right) \sin\left(r\pi \frac{x}{L}\right) dx &= \\ &= \frac{1}{2} \int_0^L \left\{ \cos\left((s-r)\pi \frac{x}{L}\right) - \cos\left((s+r)\pi \frac{x}{L}\right) \right\} dx = \frac{L}{2} \delta_{rs} \end{aligned}$$

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Energy in a string.

- Kinetic energy:

$$T = \int_0^L dT = \frac{L}{4} \rho \sum_s (\omega_s v_s \cos(\omega_s t) + \omega_s \mu_s \sin(\omega_s t))^2$$

- Potential energy:

$$U = \frac{\tau}{2} \int_0^L \left(\frac{\partial q}{\partial x}\right)^2 dx = \frac{\rho L}{4} \sum_s \omega_s^2 (\mu_s \cos(\omega_s t) - v_s \sin(\omega_s t))^2$$

- Total energy:

$$E = T + U = \frac{\rho L}{4} \sum_s \left\{ \omega_s^2 (\mu_s^2 + v_s^2) \right\} = \text{constant}$$

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ENOUGH FOR TODAY?

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