Classical Mechanics
Phy 235, Lecture 24.

Frank L. H. Wolfs
Department of Physics and Astronomy
University of Rochester

Chapter 13: Waves.

- Consider the following:
  - Increase the number of masses n to infinity.
  - Decrease the distance d to zero such that $(n + 1)d = L$.
  - Decrease the mass m to zero such that m/d = constant.
- In this case, the displacement q can be written as:

$$q_j(t) = \sum_s \beta_s \sin\left(s\pi \frac{x}{L}\right) e^{i\omega_s t} = q(x,t)$$

Wave solution.

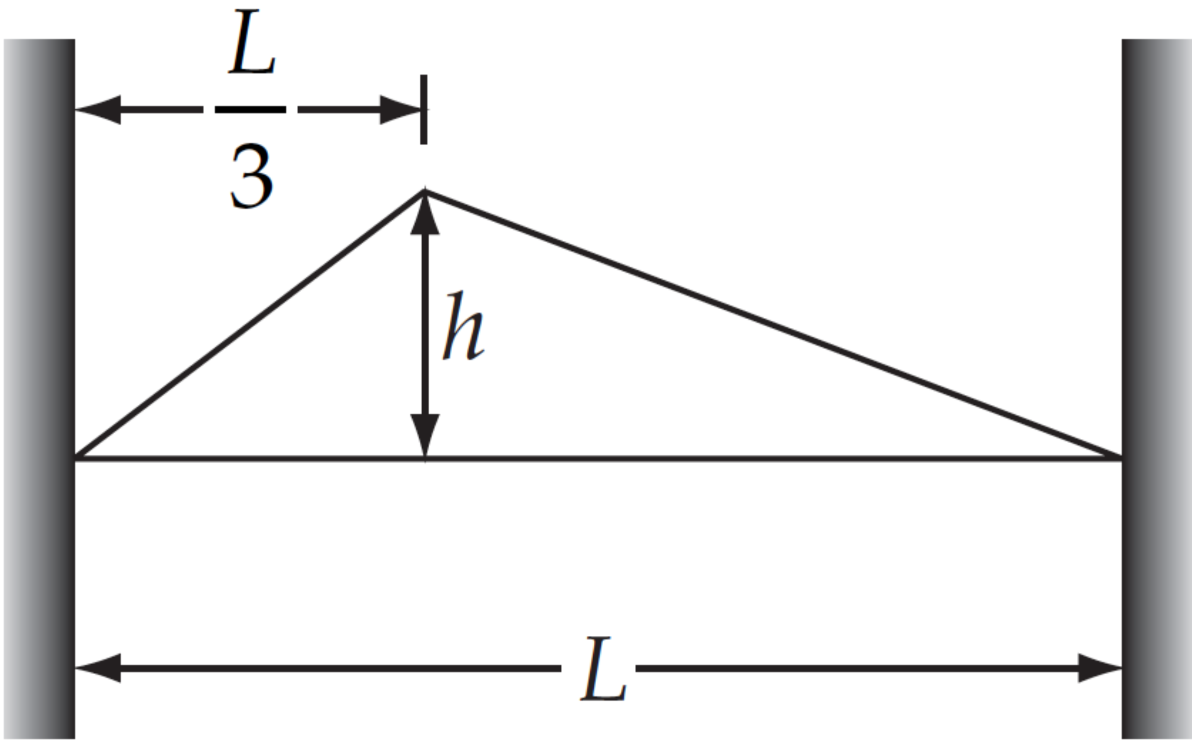
- If the displacement and velocity at $t = 0$ are known, the constants in the expression for q can be determined.
- In order to find these constants, we multiply each side by $\sin(r\pi x/L)$ and integrate x between 0 and L. We use the following fact:

$$\int_0^L \sin\left(s\pi\frac{x}{L}\right)\sin\left(r\pi\frac{x}{L}\right)dx =$$

$$= \frac{1}{2}\int_0^L \left\{\cos\left((s-r)\pi\frac{x}{L}\right) - \cos\left((s+r)\pi\frac{x}{L}\right)\right\}dx = \frac{L}{2}\delta_{rs}$$

Problem 13.2

- Rework the problem in Example 13.1 in the event that the plucked point is a distance $L/3$ from one end. Comment on the nature of the allowed modes.



Energy in a string.

- Kinetic energy:

$$T = \int_0^L dT = \frac{L}{4}\rho\sum_s \left(\omega_s v_s \cos(\omega_s t) + \omega_s \mu_s \sin(\omega_s t)\right)^2$$

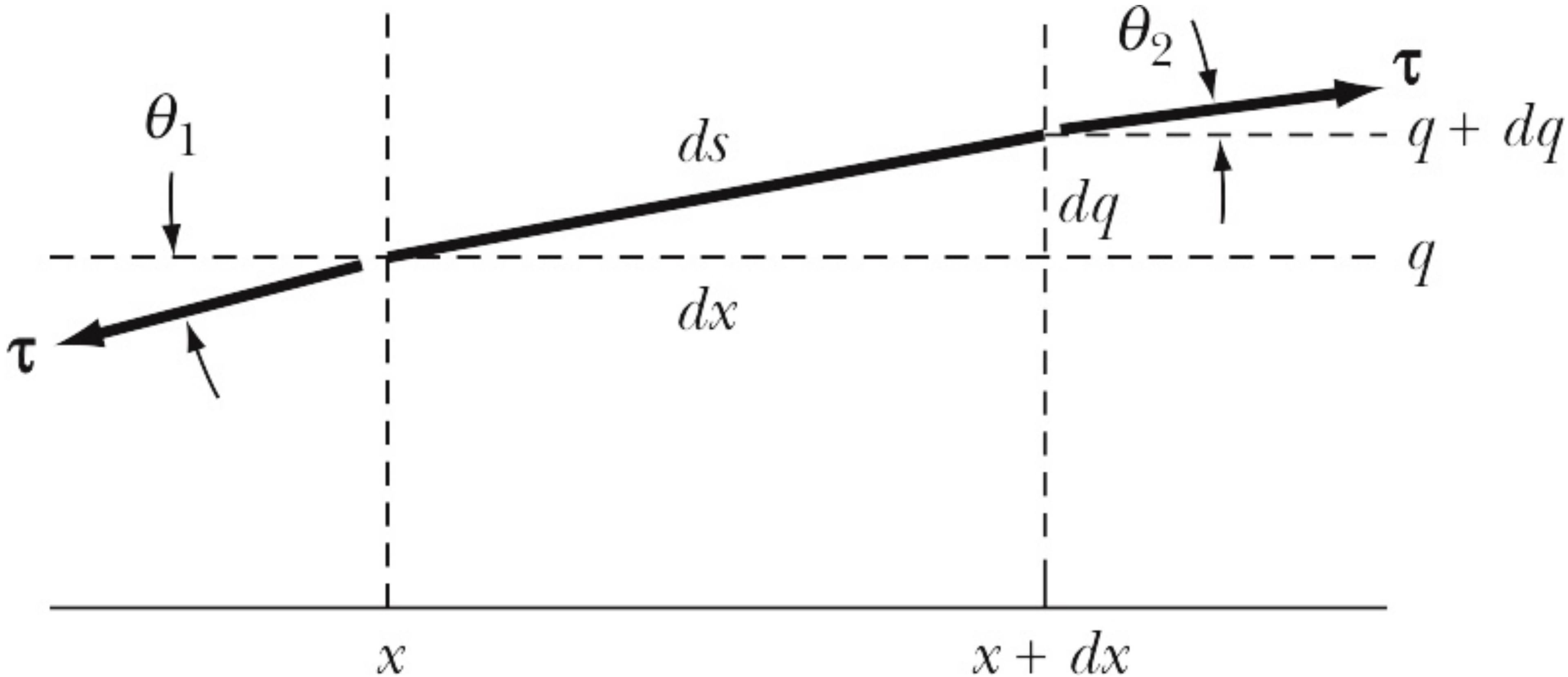
- Potential energy:

$$U = \frac{\tau}{2}\int_0^L \left(\frac{\partial q}{\partial x}\right)^2 dx = \frac{\rho L}{4}\sum_s \omega_s^{\,2}\left(\mu_s \cos(\omega_s t) - v_s \sin(\omega_s t)\right)^2$$

- Total energy:

$$E = T + U = \frac{\rho L}{4}\sum_s \left\{\omega_s^{\,2}\left(\mu_s^{\,2} + v_s^{\,2}\right)\right\} = \text{constant}$$

The wave equation.



The wave equation.

- The ideal wave equation:

$$\frac{\partial^2 q}{\partial x^2} = \frac{\rho}{\tau}\frac{\partial^2 q}{\partial t^2}$$

- The “real” wave equation:

$$\frac{\partial^2 q}{\partial x^2} - \frac{D}{\tau}\frac{\partial q}{\partial t} + \frac{F(x,t)}{\tau} = \frac{\rho}{\tau}\frac{\partial^2 q}{\partial t^2}$$

Problem 13.11.

- When a particular driving force is applied to a string, it is observed that the string vibration is purely in the n^{th} harmonic. Find the driving force.

$$f_s(t) = \int_0^L F(x,t)\sin\frac{s\pi x}{b}\,dx = 0 \quad \text{for } s \neq n$$

$$= \int_0^L F(x,t)\sin\frac{s\pi x}{b}\,dx \neq 0 \quad \text{for } s = n$$

2 Minute 27 Second Intermission.

- Since paying attention for 1 hour and 15 minutes is hard when the topic is physics, let's take a 2 minute 27 second intermission.

- You can:
  - Stretch out.
  - Talk to your neighbors.
  - Ask me a quick question.
  - Enjoy the fantastic music.

Solving the ideal wave equation.

- The ideal wave equation:

$$\frac{\partial^2 q}{\partial x^2} = \frac{\rho}{\tau}\frac{\partial^2 q}{\partial t^2}$$

- No dissipation: energy is conserved.
- Use **separation of variables** to solve the wave equation:

$$q(x,t) = \psi(x)\chi(t)$$

- This results on two differential equations:

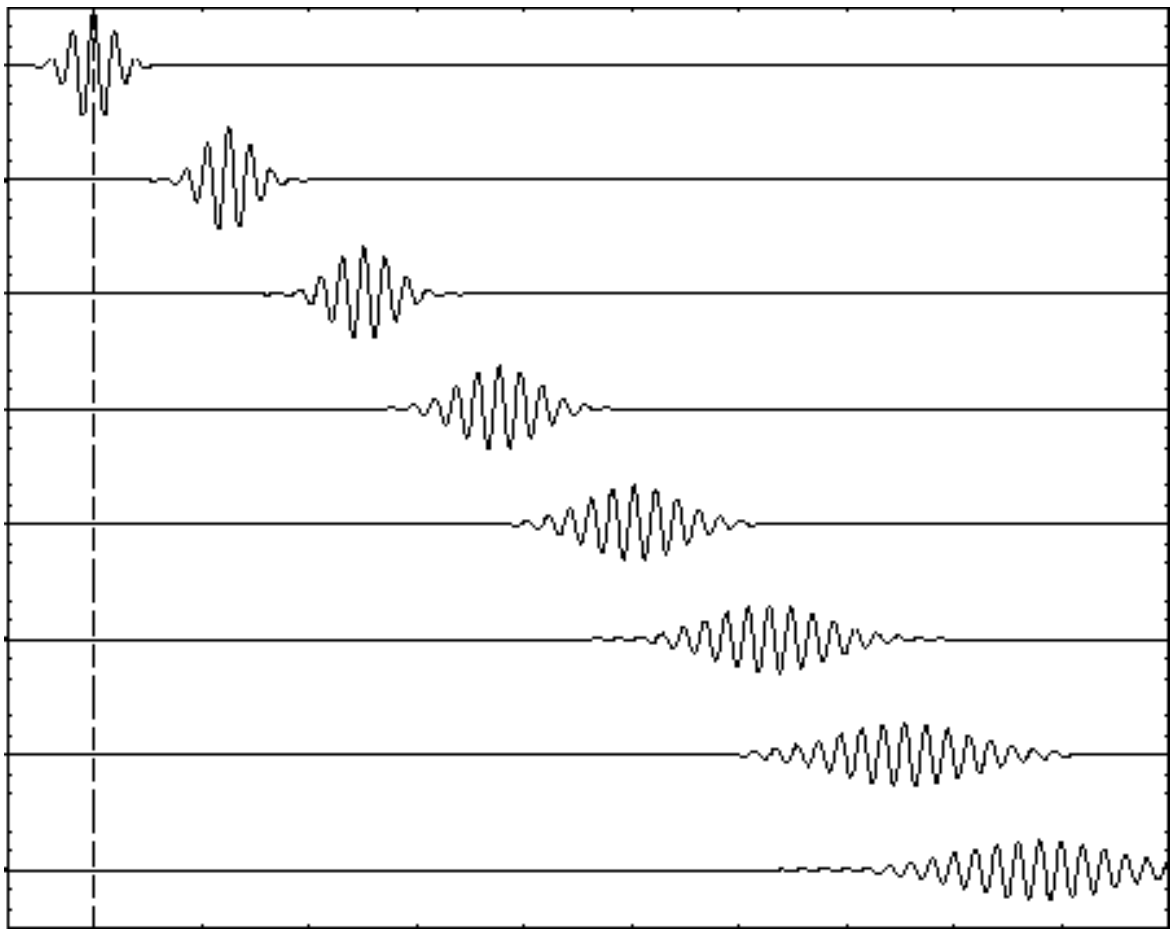
$$\frac{v^2}{\psi}\frac{\partial^2 \psi}{\partial x^2} = \omega^2 \quad \Leftrightarrow \quad \frac{\partial^2 \psi}{\partial x^2} - \frac{\omega^2}{v^2}\psi = 0$$

$$\frac{1}{\chi}\frac{\partial^2 \chi}{\partial t^2} = \omega^2 \quad \Leftrightarrow \quad \frac{\partial^2 \chi}{\partial t^2} - \omega^2 \chi = 0$$

Wave velocities

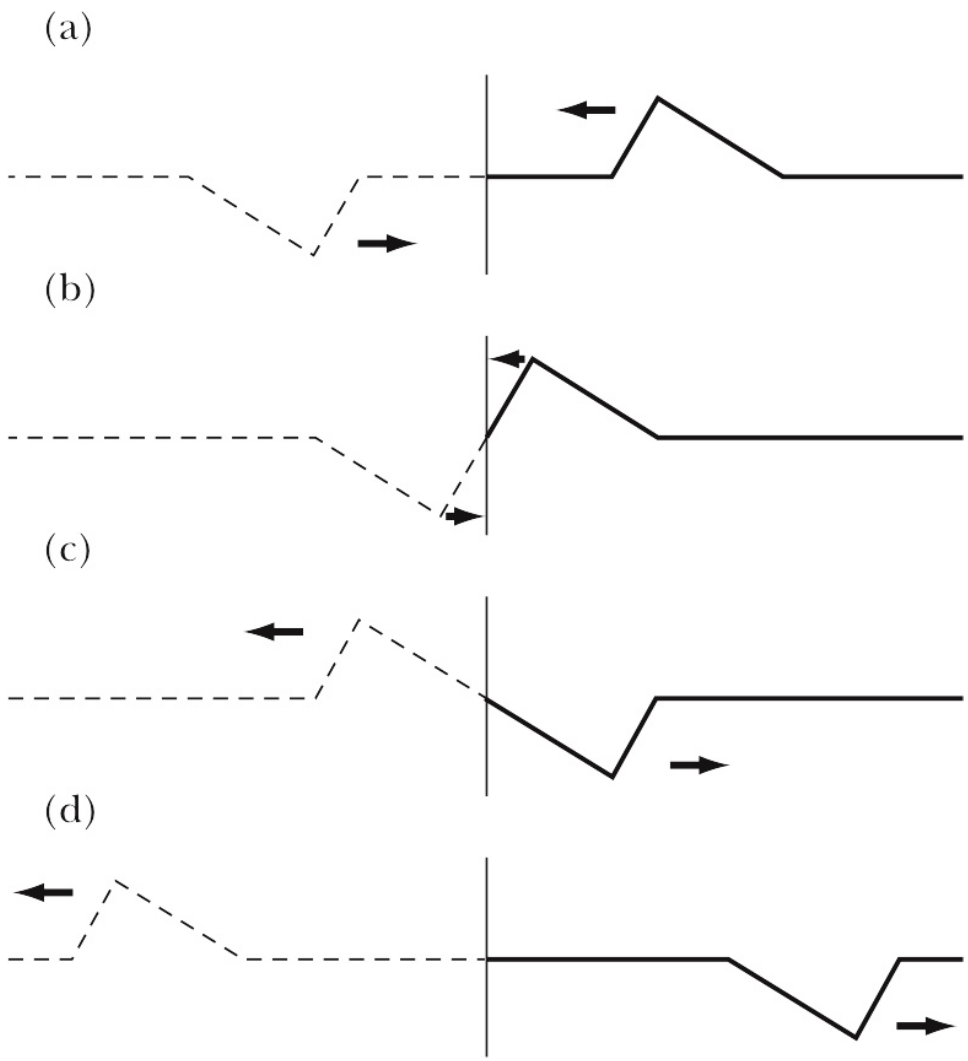
- Wave velocity:
  - Velocity that keeps the amplitude constant: v.
- Phase velocity:
  - Velocity that keeps the phase constant: V.
- The velocities depend on the wave number. When this is the case, the medium is called a **dispersive medium**.

Dispersion.
Limits information transmission.



http://jick.net/skept/GWP/

Wave Propagation: reflection at fixed end.



Other wave features.

- Two wave travelling in opposite directions can create a **standing wave**:

$$q(x,t) = A\left\{e^{-ik(x+vt)} + e^{-ik(x-vt)}\right\} = 2Ae^{-ikx}\cos(\omega t)$$

- For the loaded string:
  - There is a minimum wave length. $\lambda_n = \dfrac{2L}{n}$
  - There is a maximum wave number. $k_n = \dfrac{2\pi}{\lambda_n} = \dfrac{2\pi}{\left(\dfrac{2L}{n}\right)} \simeq \dfrac{\pi}{d}$
  - There is a maximum frequency. $\omega_n = 2\sqrt{\dfrac{\tau}{md}}\sin\left\{\dfrac{k_n d}{2}\right\} = 2\sqrt{\dfrac{\tau}{md}}$

Complex wavenumbers.

- Higher frequencies can be supported if the wavenumber becomes complex:

$$\omega = 2\sqrt{\frac{\tau}{md}}\sin\left\{\frac{d}{2}(\kappa - i\beta)\right\} =$$

$$= 2\sqrt{\frac{\tau}{md}}\left\{\sin\left(\frac{d}{2}\kappa\right)\cos\left(\frac{i\beta d}{2}\right) - \cos\left(\frac{d}{2}\kappa\right)\sin\left(\frac{i\beta d}{2}\right)\right\} =$$

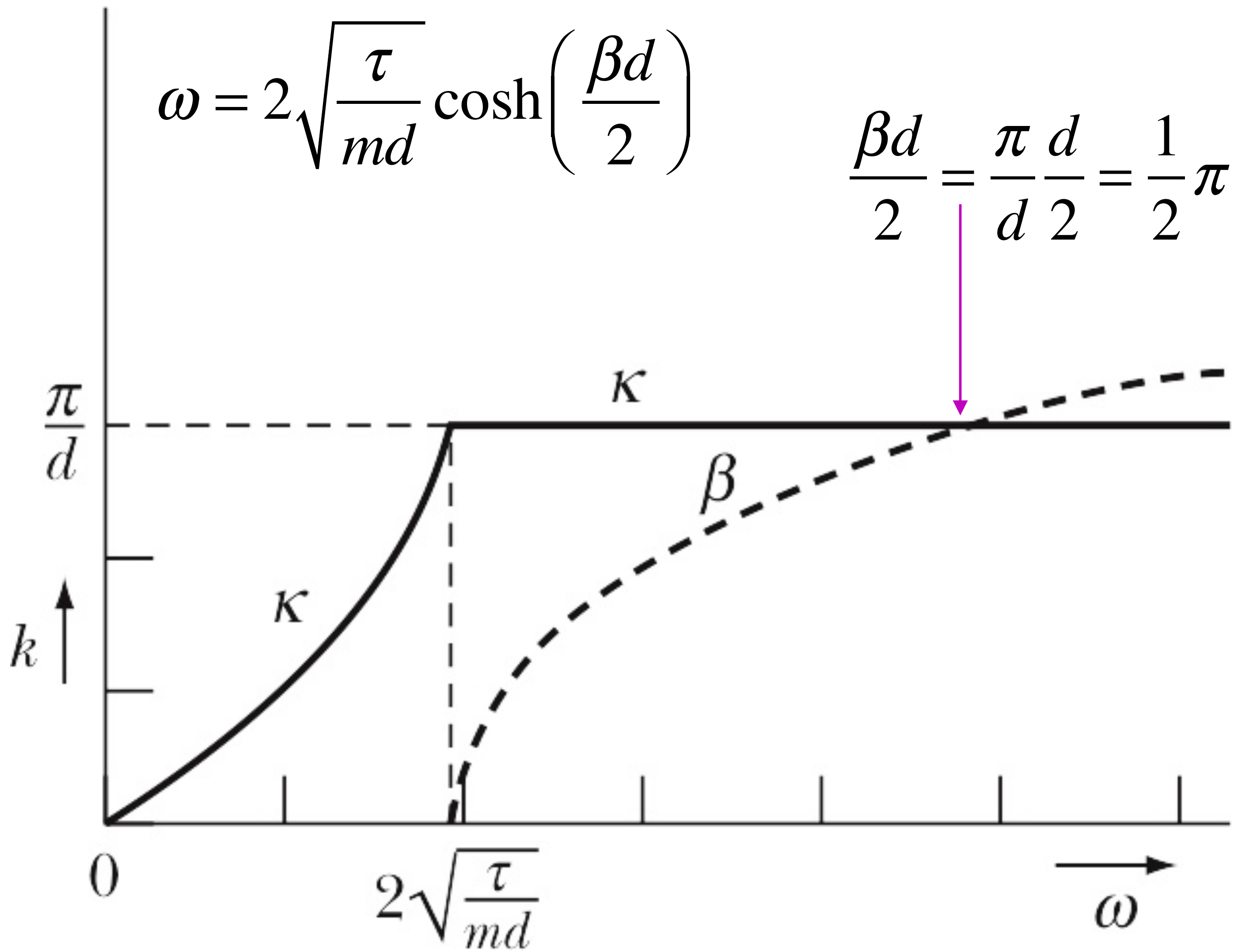
$$= 2\sqrt{\frac{\tau}{md}}\left\{\sin\left(\frac{d}{2}\kappa\right)\cosh\left(\frac{\beta d}{2}\right) - i\cos\left(\frac{d}{2}\kappa\right)\sinh\left(\frac{\beta d}{2}\right)\right\}$$

Note: the equation in my notes is missing κ in the last two steps.

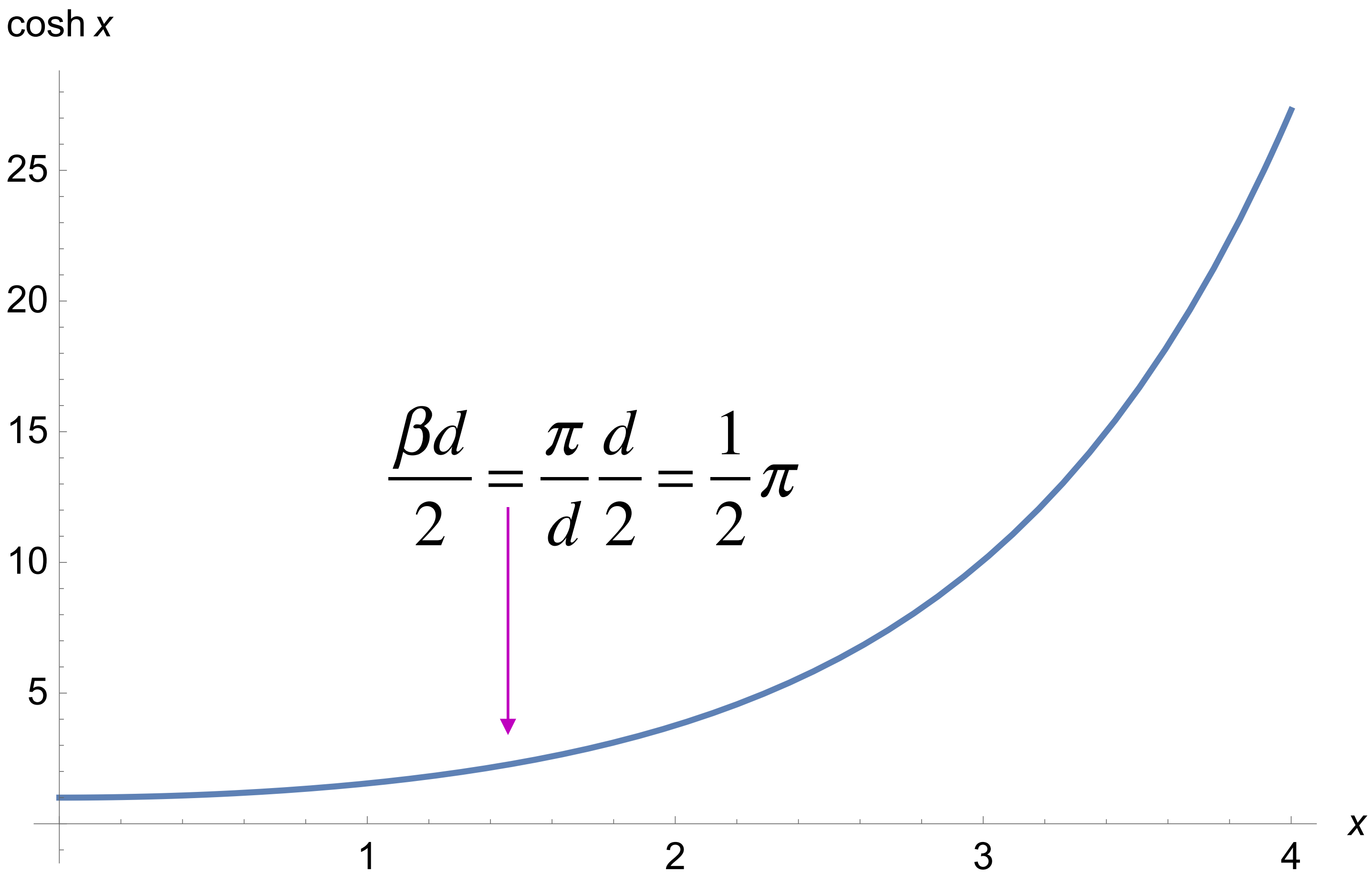
Solutions at high frequencies.

$$\omega = 2\sqrt{\frac{\tau}{md}}\cosh\left(\frac{\beta d}{2}\right)$$

$$\frac{\beta d}{2} = \frac{\pi}{d}\frac{d}{2} = \frac{1}{2}\pi$$



cosh(x) function.



$$\frac{\beta d}{2} = \frac{\pi}{d}\frac{d}{2} = \frac{1}{2}\pi$$

ENOUGH FOR TODAY?

ENOUGH FOR THIS SEMESTER?

DONE!