

Classical Mechanics Phy 235, Lecture 24.

Frank L. H. Wolfs
Department of Physics and Astronomy
University of Rochester

Frank L. H. Wolfs

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Chapter 13: Waves.

- Consider the following:
 - Increase the number of masses n to infinity.
 - Decrease the distance d to zero such that $(n + 1)d = L$.
 - Decrease the mass m to zero such that $m/d = \text{constant}$.
- In this case, the displacement q can be written as:

$$q_j(t) = \sum_s \beta_s \sin\left(s\pi \frac{x}{L}\right) e^{i\omega_s t} = q(x, t)$$

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Wave solution.

- If the displacement and velocity at $t = 0$ are known, the constants in the expression for q can be determined.
- In order to find these constants, we multiply each side by $\sin(r\pi x/L)$ and integrate x between 0 and L . We use the following fact:

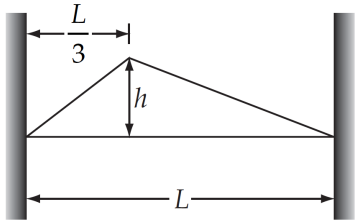
$$\begin{aligned} \int_0^L \sin\left(s\pi \frac{x}{L}\right) \sin\left(r\pi \frac{x}{L}\right) dx &= \\ &= \frac{1}{2} \int_0^L \left\{ \cos\left((s-r)\pi \frac{x}{L}\right) - \cos\left((s+r)\pi \frac{x}{L}\right) \right\} dx = \frac{L}{2} \delta_{rs} \end{aligned}$$

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Problem 13.2

- Rework the problem in Example 13.1 in the event that the plucked point is a distance $L/3$ from one end. Comment on the nature of the allowed modes.



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Energy in a string.

- Kinetic energy:

$$T = \int_0^L dT = \frac{L}{4} \rho \sum_s (\omega_s v_s \cos(\omega_s t) + \omega_s \mu_s \sin(\omega_s t))^2$$

- Potential energy:

$$U = \frac{\tau}{2} \int_0^L \left(\frac{\partial q}{\partial x} \right)^2 dx = \frac{\rho L}{4} \sum_s \omega_s^2 (\mu_s \cos(\omega_s t) - v_s \sin(\omega_s t))^2$$

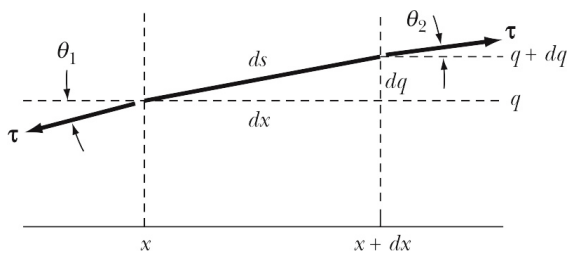
- Total energy:

$$E = T + U = \frac{\rho L}{4} \sum_s \{ \omega_s^2 (\mu_s^2 + v_s^2) \} = \text{constant}$$

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The wave equation.



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The wave equation.

- The ideal wave equation:

$$\frac{\partial^2 q}{\partial x^2} = \frac{\rho}{\tau} \frac{\partial^2 q}{\partial t^2}$$

- The "real" wave equation:

$$\frac{\partial^2 q}{\partial x^2} - \frac{D}{\tau} \frac{\partial q}{\partial t} + \frac{F(x,t)}{\tau} = \frac{\rho}{\tau} \frac{\partial^2 q}{\partial t^2}$$

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Problem 13.11.

- When a particular driving force is applied to a string, it is observed that the string vibration is purely in the n^{th} harmonic. Find the driving force.

$$\begin{aligned} f_s(t) &= \int_0^L F(x,t) \sin \frac{s\pi x}{b} dx = 0 \quad \text{for } s \neq n \\ &= \int_0^L F(x,t) \sin \frac{s\pi x}{b} dx \neq 0 \quad \text{for } s = n \end{aligned}$$

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2 Minute 27 Second Intermission.

- Since paying attention for 1 hour and 15 minutes is hard when the topic is physics, let's take a 2 minute 27 second intermission.

- You can:
 - Stretch out.
 - Talk to your neighbors.
 - Ask me a quick question.
 - Enjoy the fantastic music.



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Solving the ideal wave equation.

- The ideal wave equation:

$$\frac{\partial^2 q}{\partial x^2} = \frac{\rho}{\tau} \frac{\partial^2 q}{\partial t^2}$$

- No dissipation: energy is conserved.
- Use **separation of variables** to solve the wave equation:

$$q(x,t) = \psi(x)\chi(t)$$

- This results on two differential equations:

$$\frac{v^2}{\psi} \frac{\partial^2 \psi}{\partial x^2} = \omega^2 \Leftrightarrow \frac{\partial^2 \psi}{\partial x^2} - \frac{\omega^2}{v^2} \psi = 0$$

$$\frac{1}{\chi} \frac{\partial^2 \chi}{\partial t^2} = \omega^2 \Leftrightarrow \frac{\partial^2 \chi}{\partial t^2} - \omega^2 \chi = 0$$

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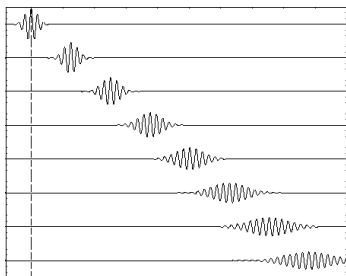
Wave velocities

- Wave velocity:
 - Velocity that keeps the amplitude constant: v .
- Phase velocity:
 - Velocity that keeps the phase constant: V .
- The velocities depend on the wave number. When this is the case, the medium is called a **dispersive medium**.

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Dispersion. Limits information transmission.

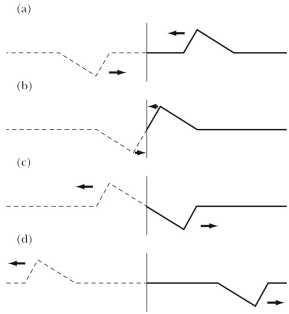


<http://fjick.net/skept/GWP/>

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Wave Propagation: reflection at fixed end.



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Other wave features.

- Two wave travelling in opposite directions can create a **standing wave**:

$$q(x,t) = A \left\{ e^{-ik(x+vt)} + e^{-ik(x-vt)} \right\} = 2Ae^{-ikx} \cos(\omega t)$$

- For the loaded string:

- There is a minimum wave length. $\lambda_n = \frac{2L}{n}$

- There is a maximum wave number. $k_n = \frac{2\pi}{\lambda_n} = \frac{2\pi}{\left(\frac{2L}{n}\right)} = \frac{\pi}{L}$

- There is a maximum frequency. $\omega_n = 2\sqrt{\frac{\tau}{md}} \sin\left\{\frac{k_n d}{2}\right\} = 2\sqrt{\frac{\tau}{md}} \sin\left\{\frac{\pi n d}{2L}\right\}$

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Complex wavenumbers.

- Higher frequencies can be supported if the wavenumber becomes complex:

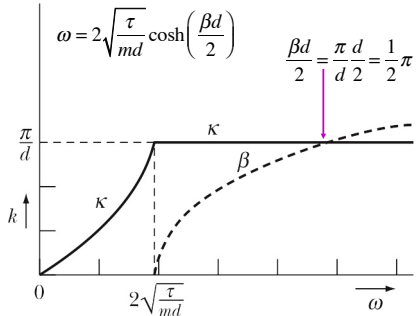
$$\begin{aligned} \omega &= 2\sqrt{\frac{\tau}{md}} \sin\left\{\frac{d}{2}(\kappa - i\beta)\right\} = \\ &= 2\sqrt{\frac{\tau}{md}} \left\{ \sin\left(\frac{d}{2}\kappa\right) \cos\left(\frac{i\beta d}{2}\right) - \cos\left(\frac{d}{2}\kappa\right) \sin\left(\frac{i\beta d}{2}\right) \right\} = \\ &= 2\sqrt{\frac{\tau}{md}} \left\{ \sin\left(\frac{d}{2}\kappa\right) \cosh\left(\frac{\beta d}{2}\right) - i \cos\left(\frac{d}{2}\kappa\right) \sinh\left(\frac{\beta d}{2}\right) \right\} \end{aligned}$$

Note: the equation in my notes is missing κ in the last two steps.

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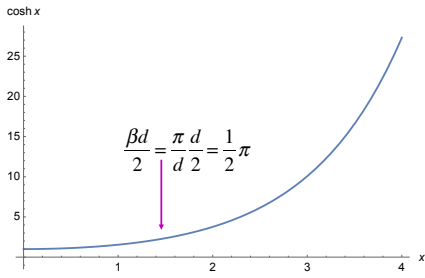
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Solutions at high frequencies.



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cosh(x) function.



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ENOUGH FOR TODAY?

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ENOUGH FOR THIS SEMESTER?

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DONE!

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